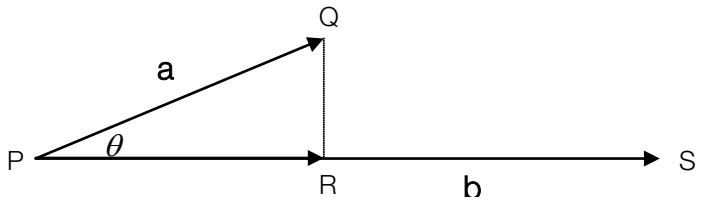
VE1.4: PROJECTION OF VECTORS

Scalar projection

Consider the diagram below:



Let $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PS} = \mathbf{b}$.

The **scalar projection** of vector **a** in the direction of vector **b** is the length of the straight line PR or $\left|\overrightarrow{PR}\right|$.

$$\left|\overrightarrow{PR}\right| = |\mathbf{a}|\cos\theta.$$

Also $\cos\theta = \dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ (because $\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$)

Therefore

$$\left|\overrightarrow{PR}\right| = (|\mathbf{a}|)\left(\frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) = \frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{b}|} = \mathbf{a}\bullet\hat{\mathbf{b}}$$ (cancel $|\mathbf{a}|$, and use $\dfrac{\mathbf{b}}{|\mathbf{b}|} = \hat{\mathbf{b}}$)

> The **scalar projection** of a vector **a** in the direction of vector **b** is given by:
>
> $$\frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{b}|} = \mathbf{a}\bullet\hat{\mathbf{b}} \quad \text{or} \quad |\mathbf{a}|\cos\theta$$

Example

Find (a) the scalar projection of
vector **a** = (2, 3, 1) in the direction of vector **b** = (5, –2, 2).

$$|\mathbf{b}| = \sqrt{25+4+4} = \sqrt{33} \quad \text{therefore} \quad \hat{\mathbf{b}} = \frac{(5,\ -2,\ 2)}{\sqrt{33}}$$

$$\mathbf{a}\bullet\hat{\mathbf{b}} = (2, 3, 1)\bullet\frac{(5,\ -2,\ 2)}{\sqrt{33}} = \frac{10-6+2}{\sqrt{33}} = \frac{6}{\sqrt{33}}$$

The scalar projection of **a** in the direction of **b** is $\dfrac{6}{\sqrt{33}}$

Vector Projection

The **vector projection** of vector **a** in the direction of vector **b** is a **vector** in the direction of **b** with **a magnitude** equal to the length of the straight line PR or $\left|\overrightarrow{PR}\right|$.

Therefore the vector projection of **a** in the direction of **b** is the scalar projection multiplied by a unit vector in the direction of **b.**

> The **vector projection** of vector **a** in the direction of vector **b** is
>
> $$(\mathbf{a}\bullet\hat{\mathbf{b}})\hat{\mathbf{b}} = \frac{(\mathbf{a}\bullet\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}$$

Examples

1. Find the vector projection of vector **a** = (2, 3, 1) in the direction of vector **b** = (5, –2, 2).

The vector projection **a** in the direction of **b** equals:

$$(\mathbf{a}\bullet\hat{\mathbf{b}})\hat{\mathbf{b}} = \frac{(\mathbf{a}\bullet\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2} \qquad |\mathbf{b}| = \sqrt{5^2+(-2)^2+2^2} = \sqrt{33}$$

$$= \frac{(2,3,1)\bullet(5,-2,2)}{33}(5,-2,2)$$

$$= \frac{6(5,-2,2)}{33}$$

The vector projection of **a** in the direction of **b** is $\dfrac{6(5,-2,2)}{33}$

2. If **a** = (1, –2, 2) and **b** = (5, –2, 2) find:

(a) The scalar projection of **a** in the direction of **b**
(b) The vector projection of **a** in the direction of **b**

(a) The scalar projection of **a** in the direction of **b** = $\mathbf{a}\bullet\hat{\mathbf{b}}$

b = (5, –2, 2)

$$|\mathbf{b}| = \sqrt{33}, \qquad \hat{\mathbf{b}} = \frac{(5,-2,2)}{\sqrt{33}}$$

$$\mathbf{a}\bullet\hat{\mathbf{b}} = (1,-2,2)\bullet\frac{(5,-2,2)}{\sqrt{33}} = \frac{13}{\sqrt{33}}$$

(b) The vector projection of **a** in the direction of **b** = $\mathbf{a}\bullet\hat{\mathbf{b}}\,(\hat{\mathbf{b}})$

$$\mathbf{a}\bullet\hat{\mathbf{b}} = \frac{13}{\sqrt{33}}$$ (from part (a))

$$\mathbf{a} \bullet \hat{\mathbf{b}}\,(\hat{\mathbf{b}}) = \frac{13}{\sqrt{33}} \frac{(5,-2,2)}{\sqrt{33}} = \frac{13(5,-2,2)}{33}$$

The vector projection of **a** in the direction of **b** = $\dfrac{13(5,-2,2)}{33}$

Exercises

Exercise 1 For **a** = (2, 3 ,1), **b** = (5, 0, 3), **c** = (0, 0, 3) and **d** = (–2, 2, –1) find:

(a) the scalar projection of **a** in the direction of **b**

(b) the vector projection of **a** in the direction of **b**

(c) the scalar projection of **c** in the direction of **b**

(d) the vector projection of **c** in the direction of **a**

(e) the vector projection of **d** in the direction of **a**

(f) the vector projection of **b** in the direction of **d**

Exercise 2 If **a** = (2, 0,-1) and **b** = (3, 5, 6)

(a) find the scalar projection of **a** in the direction of **b.**

(b) what can you say about the relationship between **a** and **b?**

Answers

Exercise 1.

(a) $\dfrac{13}{\sqrt{34}}$, (b) $\dfrac{13}{34}(5,0,3)$, (c) $\dfrac{9}{\sqrt{34}}$, (d) $\dfrac{3}{14}(2,3,1)$

(e) $\dfrac{1}{14}(2,3,1)$ (f) $\dfrac{-13}{9}(-2,2,-1)$,

Exercise 2.

(a) 0. (b) a and b are perpendicular.