VE1.2: SCALAR PRODUCT

Definition

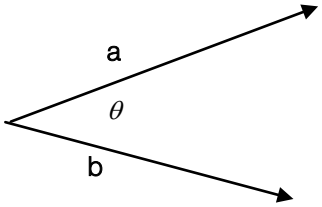
The **scalar,** or **dot,** product of two vectors **a** (a_1, a_2, a_3) and **b** (b_1, b_2, b_3) is a **scalar**, defined by:

$$\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

or geometrically

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where θ is the angle between **a** and **b**.



Properties

1. If **a** and **b** are non-zero vectors and **a** is perpendicular to **b** then $\mathbf{a} \bullet \mathbf{b} = 0$, (cos(π/2)= 0).

2. If **a** is parallel to **b** then $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|$ (cos(0)=1)

3. The dot product does not depend on the order of multiplication.

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$$

Therefore

$$\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0$$

$$\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$$

Examples

(1) (2**i** + 3**j** + 4**k**) • (– **i** – 2**j** + **k**) = (2 × –1) + (3 × – 2) + (4 × 1) = – 4
(2) (2, – 3, – 3) • (1, 1, – 2) = 2 – 3 + 6 = 5
(3) (5, 0, – 1) • (1, 4, 3) = 5 + 0 – 3 = 2
(4) (2**i** + 4**k**) • (– 3**i** – 2**j**) = –6 + 0 + 0 = – 6
See *Exercises 1, 2, and 3.*

Angle between two vectors

The angle, θ, $(0 \leqslant \theta \leq \pi)$, between two vectors can be found using the definition of the dot product.

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

therefore $\cos\theta = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ and

$$\theta = \cos^{-1} \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \quad \text{or} \quad \theta = \cos^{-1} \frac{\boldsymbol{a} \bullet \hat{\boldsymbol{b}}}{|\boldsymbol{a}|}$$

Examples

1. If **a** = (2, 3, 1) and **b** = (5, –2, 2) find the angle θ, between **a** and **b**.

$$\theta = \cos^{-1} \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$\mathbf{a} \bullet \mathbf{b}$ = (2, 3, 1) • (5, –2, 2) = 6

$|\mathbf{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

$|\mathbf{b}| = \sqrt{25 + 4 + 4} = \sqrt{33}$

$$\therefore \quad \theta = \cos^{-1} \frac{6}{\sqrt{33} \times \sqrt{14}} = \cos^{-1} 0.2791$$

$$\therefore \theta = 74°$$

The angle between **a** and **b** is 74^0.

2. Find the angle θ, between **a,** (1, 0, 1) and **b,** (–2, –1, 1).

$\mathbf{a} \bullet \mathbf{b}$ = (1, 0, 1) • (–2, –1 , 1) = –1

$|\mathbf{a}| = \sqrt{2}$ $\quad$ $|\mathbf{b}| = \sqrt{6}$

$$\therefore \quad \theta = \cos^{-1} \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \cos^{-1} \frac{-1}{\sqrt{2} \times \sqrt{6}} = \cos^{-1}(-0.2887)$$

$$\therefore \theta = 106.8^0$$

The angle between **a** and **b** is 106.8^0

See *Exercises 4 and 5.*

Exercises

Exercise 1.

Calculate the dot product of:

(a) (2, 5, –1) and (4, 1,1) (b) $3\mathbf{i}$ and $5\mathbf{j}$ (c) $5\mathbf{k}$ and $(\mathbf{j} + 2\mathbf{k})$

Exercise 2.

Find:

(a) (2, 0, 4) • (– 3, 1, 3) (b) (0, 5, 1) • (4, 0, 0) (c) $(2\mathbf{i} + 3\mathbf{k}) \bullet (7\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$

Exercise 3.

Which of the following vectors are perpendicular?

(i) (5, 2, 3), (ii) (0,1, –1), (iii) (– 2, 2, 2)

Exercise 4.

Find the angle between the following pairs of vectors:

(a) (1, 2, 3) and (4, –1, 0) (b) (2, 1, –2) and (1,5, –1)
(c) (0, 5, 1) and (2,0,0) (d) (1, –2, 3) and (–4, 1, –3)
(e) (2, 1, –2) and (0,4,0) (f) (0, 3 0) and (0, 1, 0)

Exercise 5.

If $\mathbf{a}$ = (2, 2 ,2), $\mathbf{b}$ = (3, 2, –1), and $\mathbf{c}$ = (– 1, 4, 1), find:

(a) Show $\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{c}$

(b) Rearranging $\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{c}$ gives $\mathbf{a} \bullet (\mathbf{b} - \mathbf{c}) = 0$. As $\mathbf{b} \neq \mathbf{c}$ what is the relationship between $\mathbf{a}$ and $(\mathbf{b} - \mathbf{c})$?

Answers

1. (a) 12 (b) 0 (c) 10

2. (a) 6 (b) 0 (c) 26

3. (i) and (iii) , (ii) and (iii)

4. (a) 82.6^0, (b) 54.7^0, (c) 90^0, (d) 141.8^0, (e) 70.5^0, (f) 0^0

5. (a) $\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{c} = 8$ (b) a is perpendicular to $(\mathbf{b} - \mathbf{c})$.