VE1.1: VECTORS: INTRODUCTION

Definition

A **vector** is a quantity that has a **magnitude** and a **direction**.

One example of a vector is velocity. The velocity of an object is determined by the magnitude (speed) and direction of travel. Other examples of vectors are force, displacement and acceleration.

A **scalar** is a quantity that has **magnitude** only. Mass, time and volume are all examples of scalar quantities.

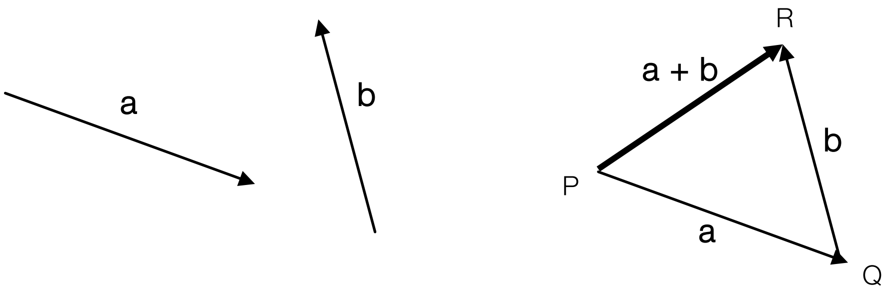
Vectors in three-dimensional space are defined by three mutually perpendicular directions and will be denoted by lower case **bold** letters such as **a, b, c**

A vector in the **opposite direction** from **a** is denoted by $-$**a**.

Vectors can be added or subtracted graphically using the triangle rule.

Adding and subtracting vectors

Triangle Rule

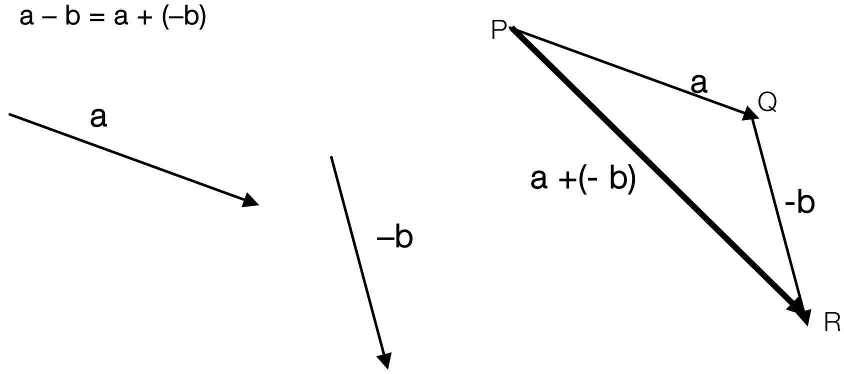


To add vectors **a** and **b** shown above place the tail of vector **b** at the head of vector **a** (point Q).

The vector sum, **a** + **b**, is the vector $\overrightarrow{PR}$, from the tail of vector **a** to the head of **b**.

To subtract **b** from **a**, reverse the direction of **b** to give –**b** then add **a** and –**b**.

a – **b** = **a** + (–**b**)



Vector $\overrightarrow{PR}$ is vector the vector **a** – **b**.

Components of a vector

In the diagram below the vector **r** is represented by $\overrightarrow{OP}$ where P is the point (x, y, z).

If **i, j** and **k** are vectors of magnitude **one** parallel to the positive directions of the x- axis, y- axis and z- axis respectively, then:

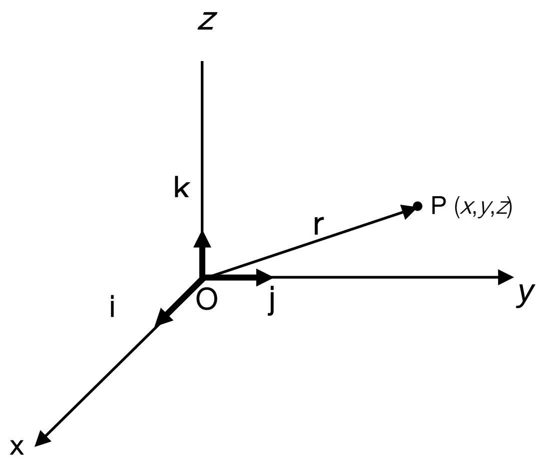
x**i** is a vector of length x in the direction of the x- axis

y**j** is a vector of length y in the direction of the y- axis

z**k** is a vector of length z in the direction of the z- axis

$\overrightarrow{OP}$ is then the vector x**i** + y**j** + z**k**.

x, y and z are called the **components** of the vector.



The notation (x, y, z) will be used to denote the vector (x**i** + y**j** + z**k**) as well as the co-ordinates of a point P (x, y, z). The context will determine the correct meaning.

Vectors may also be added or subtracted by adding or subtracting their corresponding components.

Example

If **a** = (-3, 4, 2) and **b** = (–1 –2, 3), find:

(i) **a** + **b** (ii) **a** –**b**.

Adding or subtracting components

(i) **a** + **b** = (–3, 4, 2) + (–1, –2, 3) = (–3 +(–1), 4 +(–2), 2+3) = (–4, 2, 5)

Similarly (ii) **a** –**b** = (–3, 4, 2) – (–1, –2, 3) =(–3 –(–1), 4–(–2), 2–3) = (–2, 6, –1)

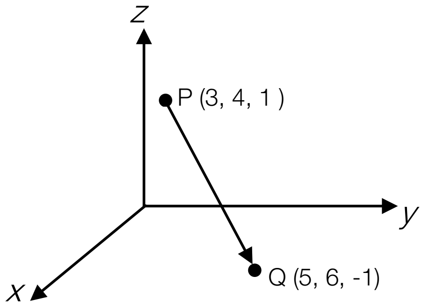
See *Exercise 1.*

Directed line segment

The directed line segment, or geometric vector, $\overrightarrow{PQ}$, from P(x_1, y_1, z_1) to Q(x_2, y_2, z_2) is found by subtracting the co-ordinates of P (the initial point) from the co-ordinates of Q (the final point).

$$\overrightarrow{PQ} = \left((x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}\right)$$

Example



$$\overrightarrow{PQ} = (5-3)\mathbf{i} + (6-4)\mathbf{j} + (-1-1)\mathbf{k}$$

$$\overrightarrow{PQ} = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

The directed line segment $\overrightarrow{PQ}$ is represented by the vector 2**i** + 2**j** – 2**k**, or

(2, 2, –2). Any other directed line segment with the **same length and same directon as** $\overrightarrow{PQ}$ is also represented by 2**i** + 2**j** – 2**k** or (2, 2, –2).

The directed line segment $\overrightarrow{QP}$ has the same length as $\overrightarrow{PQ}$ but is in the opposite direction.

$\overrightarrow{QP} = -\overrightarrow{PQ}$ = – (2**i** + 2**j** – 2**k**) = – 2**i** – 2**j** + 2**k** or (–2,–2,2)

Position vector

The **position vector** of any point is the directed line segment from the **origin O** (0,0,0) to that point and is given by the co-ordinates of of the point.

The position vector of P(3, 4, 1) is 3**i** + 4**j** + **k**, or (3, 4, 1).

See *Exercise 2.*

Magnitude of a vector

If **a** = a_1**i** + a_2**j** + a_3**k** the length or magnitude of **a** is written $|\mathbf{a}|$ or 'a' and is evaluated as:

$$|\boldsymbol{a}| = \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}$$

Example

The length of the vector 2**i** + 3**j** - 5**k** equals $\sqrt{2^2 + 3^2 + (-5)^2} = \sqrt{38}$

a_1, a_2 , a_3 are referred to as the **components** of vector **a**.

Unit vector

Any vector with a magnitude of **one** is called a **unit vector**.

If **a** is any vector then a unit vector parallel to **a** is written $\hat{\mathbf{a}}$ (a "hat"). The "hat" symbolises a unit vector.

The unit vector $\hat{\mathbf{a}}$ equals vector **a** divided by its magnitude $|\mathbf{a}|$

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Or **a** = $|\mathbf{a}|\hat{\mathbf{a}}$

Of particular importance are unit vectors **i**, **j** and **k**, parallel to the *x*-,*y*-, and *z*- axes respectively.

Examples

1. If $\overrightarrow{PQ}$ is the line 2**i** −5**j** + **k** find a unit vector parallel to $\overrightarrow{PQ}$

$$|\overrightarrow{PQ}| = \sqrt{2^2 + (-5)^2 + 1^2} = \sqrt{30}$$

A unit vector parallel to $\overrightarrow{PQ}$ is $\dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \widehat{PQ}$

$$\widehat{PQ} = \frac{1}{\sqrt{30}}(2\mathbf{i} - 5\mathbf{j} + \mathbf{k})$$

2. If **a** = (1, 2, 3) a unit vector parallel to **a** is:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{(1,\ 2,\ 3)}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{14}}(1,\ 2,\ 3)$$

See *Exercise 3.*

Multiplication by a scalar

To multiply vector **a** = $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ by a scalar, *m*, multiply each component of **a** by *m*.

$$m\mathbf{a} = ma_1\mathbf{i} + ma_2\mathbf{j} + ma_3\mathbf{k}$$

The result is a vector of length $m \times |\mathbf{a}|$

If $m > 0$ the resultant vector is in the same direction as **a**

If $m < 0$ the resultant vector is in the opposite direction from **a**.

Two vectors a and b are said to be parallel if and only if a = *k*b where *k* is a real constant.

Example

Multiply **a** = (3**i** + **j** – 2**k**) by 7 and show $|7\mathbf{a}| = 7|\mathbf{a}|$

7**a** = 7(3**i** + **j** – 2**k**)

= 21**i** + 7**j** – 14**k**.

$$|7\mathbf{a}| = \sqrt{21^2 + 7^2 + (-14)^2} = \sqrt{686} = 7\sqrt{14}$$

The magnitude of **a**, $|\mathbf{a}|$ is $\sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$

$$7|\mathbf{a}| = 7\sqrt{14}$$

therefore $|7\mathbf{a}| = 7|\mathbf{a}| = 7\sqrt{14}$

See ***Exercise 4.***

Exercises

Exercise 1

Given **a** = (2,1,1), **b** = (1,3,–3) and **c** = (0,3, –2) find:

(a) **a** + **b**, (b) **a** + **c**, (c) **c** – **b**, (d) **a** – **b**

Exercise 2

(a) Given the points A(3, 0, 4), B(–2, 4, 3), and C(1,–5, 0), find:

(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{AC}$ (iii) $\overrightarrow{CB}$

(iv) $\overrightarrow{BC}$ (v) $\overrightarrow{CA}$

Compare your answers to (ii) and (v), and also to (iii) and (iv). What do you notice?

(b) What are the position vectors of **A**, **B** and **C**.

Exercise 3

(a) Find the length of the vectors

(i) (3, – 1, – 1), (ii) (0, 2, 4) (iii) (0, –2, 0)

(b) Given **A**(3, 0, 4) **B**(0, 4, 3) and **C**(1,–5, 0),

find unit vectors parallel to (i) $\overrightarrow{BA}$ (ii) $\overrightarrow{CB}$ (iii) $\overrightarrow{AC}$

Exercise 4

(a) Find the following

(i) 3 (i + 3j –5k) (ii) – 4 (j – 3k)

(b) If **a** = (2, –2, 1), **b** = (0, 1, 1) and **c** = (–1, 3, –2) find:

(i)(2**a** + 3**b**) (ii)(3**a** –2**b**) (iii)(2**a** – **b** + 2**c**) (iv) a unit vector parallel to 2**a** – **b**

(c) Write down a vector three times the length of (6i + 2j –5k) and in the opposite direction.

Answers

Exercise 1

(a) (3, 4, −2) (b) (2, 4, −1) (c) (−1, 0, 1) (d) (1, −2, 4)

Exercise 2

(a) (i) (-5, 4, -1) (ii) (-2, -5, -4) (iii) (-3, 9, 3) (iv) (3, -9, -3)
(v) (2, 5, 4)

$\overrightarrow{AC} = -\overrightarrow{CA}$ $\overrightarrow{BC} = -\overrightarrow{CB}$

(b) $\overrightarrow{OA} = 3\mathbf{i} + 4\mathbf{k}$, $\overrightarrow{OB} = -2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OC} = \mathbf{i} - 5\mathbf{j}$

Exercise 3

(a) (i) $\sqrt{11}$ (ii) $\sqrt{20} = 2\sqrt{5}$ (iii) 2

(b) (i) $\frac{(3,-4,1)}{\sqrt{26}}$ (ii) $\frac{(-1,9,3)}{\sqrt{91}}$ (iii) $\frac{(-2,-5,-4)}{\sqrt{45}} = \frac{(-2,-5,-4)}{3\sqrt{5}}$

Exercise 4

(a) (i) (3**i** +9**j** −15**k**) (ii) (−4**j** +12**k**)

(b) (i) (4, −1,5) (ii) (6, −8,1) (iii) (2,1, −3) (iv) $\frac{(4,-5,1)}{\sqrt{42}}$

(c) (−18**i** −6**j** +15**k**)