RMIT UNIVERSITY

# IL1.5: SURDS

## Definition

A ***surd*** is an irrational number resulting from a radical expression that **cannot** be evaluated directly.

For example $\sqrt{2}$, $\sqrt{3}$, $\frac{1}{\sqrt{8}}$, $\sqrt{20}$, $\sqrt[3]{2}$, $\sqrt[3]{9}$, $\sqrt[4]{8}$, etc are all surds

but $\sqrt{1}$, $\sqrt{9}$, $\frac{1}{\sqrt{100}}$, $\sqrt[3]{8}$, $\sqrt[4]{16}$, $\sqrt[3]{64}$ are not.

Surds *cannot* be expressed exactly in the form $\frac{a}{b}$, and can only be approximated by a decimal.

eg: $\sqrt{2} \approx 1.414$

*See Exercise 1*

## Simplifying Surds

If a surd has a square factor ie 4, 9, 16, 25, 36…, then it can be simplified.

### Examples

1. $\sqrt{200} = \sqrt{100 \times 2}$
$= \sqrt{100} \times \sqrt{2}$ [NB: It is best to find the *largest* square number factor!]
$= 10\sqrt{2}$

2. $3\sqrt{48} = 3\sqrt{16 \times 3}$
$= 3\sqrt{16} \times \sqrt{3}$
$= 3 \times 4 \times \sqrt{3}$
$= 12\sqrt{3}$

3. $\frac{\sqrt[3]{24}}{4} = \frac{\sqrt[3]{8 \times 3}}{4}$
$= \frac{\sqrt[3]{8} \times \sqrt[3]{3}}{4}$
$= \frac{2 \times \sqrt[3]{3}}{4}$
$= \frac{\sqrt[3]{3}}{2}$

*See Exercise 2*

## Addition and Subtraction

Only like surds can be added or subtracted

### Examples

1. $8\sqrt{5} - 3\sqrt{5} = 5\sqrt{5}$

2. $2\sqrt{3} + 5\sqrt{7} - 10\sqrt{3} = 5\sqrt{7} - 8\sqrt{3}$

Sometimes it is necessary to simplify each surd first

3. $\sqrt{18} - \sqrt{8} - \sqrt{20} = \sqrt{9\times 2} - \sqrt{4\times 2} - \sqrt{4\times 5}$

$$= 3\sqrt{2} - 2\sqrt{2} - 2\sqrt{5}$$

$$= \sqrt{2} - 2\sqrt{5}$$

NB: $\sqrt{5} + \sqrt{7} \neq \sqrt{12}$ because $\sqrt{5}$ and $\sqrt{7}$ are not 'like' surds.

*See Exercise 3*

## Multiplication

To multiply surds the whole numbers are multiplied together, as are the numbers enclosed by the radical sign

ie

$$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$$

### Examples:

1. $\sqrt{2} \times \sqrt{3} = \sqrt{6}$

2. $3\sqrt{3} \times 4\sqrt{5} = 12\sqrt{15}$

Sometimes it is possible to simplify after multiplication

3. $2\sqrt{10} \times 7\sqrt{6} = 14\sqrt{60}$

$$= 14\sqrt{4\times 15}$$

$$= 14\times 2\sqrt{15}$$

$$= 28\sqrt{15}$$

*See Exercise 4*

# Expansion of Brackets

The usual algebraic rules for expansion of brackets apply to brackets containing surds

a(b + c) = ab + ac

and

(a + b)(c + d) = ac + bc +ad + bd

## Examples

1. $\sqrt{2}\left(\sqrt{2}+5\right)=2+5\sqrt{2}$

2. $2\sqrt{3}\left(\sqrt{3}-3\sqrt{2}\right)=2\sqrt{9}-6\sqrt{6}$

$= 6 - 6\sqrt{6}$

3. $(7-\sqrt{5})^2 = (7-\sqrt{5})(7-\sqrt{5})$

$= 49-7\sqrt{5}-7\sqrt{5}+5$

$= 54-14\sqrt{5}$

4. $\left(\sqrt{6}-4\sqrt{3}\right)\left(2\sqrt{2}-3\sqrt{5}\right) = 2\sqrt{12}-8\sqrt{6}-3\sqrt{30}+12\sqrt{15}$

$= 2\sqrt{4\times 3}-8\sqrt{6}-3\sqrt{30}+12\sqrt{15}$

$= 2\times 2\times\sqrt{3}-8\sqrt{6}-3\sqrt{30}+12\sqrt{15}$

$= 4\sqrt{3}-8\sqrt{6}-3\sqrt{30}+12\sqrt{15}$

*See Exercise 5*

# Fractions containing surds

Surds expressed as fractions may be simplified using

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

## Examples

1. $\frac{\sqrt{18}}{\sqrt{6}} = \sqrt{\frac{18}{6}} = \sqrt{3}$

2. $\frac{4}{\sqrt{2}} = \frac{\sqrt{16}}{\sqrt{2}} = \sqrt{\frac{16}{2}} = \sqrt{8}$ or $2\sqrt{2}$

*See Exercise 6*

## Rationalizing surds

Rationalized surds are expressed with a *rational denominator.*

### Examples

1. $\dfrac{2}{\sqrt{5}} = \dfrac{2}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$

   $= \dfrac{2\sqrt{5}}{5}$

2. $\dfrac{\sqrt{5}}{3\sqrt{2}} = \dfrac{\sqrt{5}}{3\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

   $= \dfrac{\sqrt{10}}{6}$

## Conjugate surds

The pair of expressions $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are called *conjugate surds.* Each is the conjugate of the other*.*

The product of two conjugate surds does **NOT** contain any surd term!

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = a - b$$

Eg: $\left(\sqrt{10}-\sqrt{3}\right)\left(\sqrt{10}+\sqrt{3}\right) = \left(\sqrt{10}\right)^2 - \left(\sqrt{3}\right)^2$

$= 10 - 3$

$= 7$

We make use of this property of conjugates to rationalize denominators of the form

$\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ .

### Example

$\dfrac{\sqrt{3}}{5+\sqrt{2}} = \dfrac{\sqrt{3}}{5+\sqrt{2}} \times \dfrac{5-\sqrt{2}}{5-\sqrt{2}}$

$= \dfrac{\sqrt{3}\left(5-\sqrt{2}\right)}{25-2}$

$= \dfrac{5\sqrt{3}-\sqrt{6}}{23}$

*See Exercise 7*

# Operations on fractions that contain surds

When adding and subtracting fractions containing surds it is generally advisable to first rationalize each fraction:

## Example

$$\frac{2}{3\sqrt{2}+1}+\frac{1}{\sqrt{3}-\sqrt{2}} = \frac{2}{3\sqrt{2}+1}\times\frac{3\sqrt{2}-1}{3\sqrt{2}-1}+\frac{1}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

$$= \frac{6\sqrt{2}-2}{18-1}+\frac{\sqrt{3}+\sqrt{2}}{3-2}$$

$$= \frac{6\sqrt{2}-2}{17}+\frac{\sqrt{3}+\sqrt{2}}{1}$$

$$= \frac{6\sqrt{2}-2}{17}+\frac{17(\sqrt{3}+\sqrt{2})}{17}$$

$$= \frac{6\sqrt{2}-2+17\sqrt{3}+17\sqrt{2}}{17}$$

$$= \frac{23\sqrt{2}-2+17\sqrt{3}}{17}$$

*See Exercise 8*

# Exercises

### Exercise 1
Decide whether the following radical expressions are rational or irrational and evaluate exactly if rational:

(a) $\sqrt{25}$ (b) $\sqrt{2.25}$ (c) $\sqrt{\frac{1}{10}}$ (d) $\sqrt[3]{64}$ (e) $\sqrt{12\times18}$

### Exercise 2
Simplify

(a) $\sqrt{20}$ (b) $\sqrt{48}$ (c) $\sqrt{500}$ (d) $2\sqrt{12}$ (e) $\frac{2}{\sqrt{20}}$

### Exercise 3
Simplify

(a) $2\sqrt{3}+5\sqrt{3}$ (b) $3\sqrt{7}+5\sqrt{7}-3\sqrt{7}$

(c) $\sqrt{2}+\sqrt{7}+3\sqrt{2}-4\sqrt{7}$ (d) $\sqrt{54}-\sqrt{24}$

### Exercise 4
Simplify

(a) $\sqrt{3}\times\sqrt{10}$ (b) $2\sqrt{7}\times5\sqrt{3}$

(c) $\sqrt{10}\times3\sqrt{10}$ (d) $2\sqrt{8}\times2\sqrt{50}\times\sqrt{2}$

### Exercise 5
Expand the brackets and simplify if possible

(a) $\sqrt{2}\left(\sqrt{2}-8\right)$ (b) $\left(2+\sqrt{3}\right)\left(\sqrt{3}-4\right)$

(c) $\left(1+\sqrt{10}\right)^2$ (d) $\left(\sqrt{11}+3\right)\left(\sqrt{11}-3\right)$

Exercise 6
Simplify

(a) $\dfrac{\sqrt{12}}{\sqrt{6}}$  (b) $\dfrac{\sqrt{32}}{\sqrt{2}}$

(c) $\dfrac{3\sqrt{28}}{\sqrt{7}}$  (d) $\dfrac{4\sqrt{2}\times 3\sqrt{3}}{6\sqrt{6}}$

Exercise 7
Express the following fractions with a rational denominator in simplest form:

1. (a) $\dfrac{\sqrt{5}}{\sqrt{2}}$  (b) $\dfrac{1}{\sqrt{10}}$  (c) $\dfrac{2\sqrt{18}}{\sqrt{8}}$

2. (a) $\dfrac{2+\sqrt{3}}{\sqrt{2}}$  (b) $\dfrac{1}{\sqrt{3}-\sqrt{2}}$  (c) $\dfrac{\sqrt{3}+2}{\sqrt{3}-2}$

Exercise 8

Evaluate and express with a rational denominator $\dfrac{2}{\sqrt{3}-1}+\dfrac{3}{2-\sqrt{3}}$

## Answers

Exercise 1
(a) 5 rational (b) 1.5 rational (c) irrational (d) 4 rational (e) irrational

Exercise 2

(a) $2\sqrt{5}$  (b) $4\sqrt{3}$  (c) $10\sqrt{5}$  (d) $4\sqrt{3}$  (e) $\dfrac{1}{\sqrt{5}}$

Exercise 3
(a) $7\sqrt{3}$  (b) $5\sqrt{7}$  (c) $4\sqrt{2} - 3\sqrt{7}$  (d) $\sqrt{6}$

Exercise 4
(a) $\sqrt{30}$  (b) $10\sqrt{21}$  (c) 30  (d) $80\sqrt{2}$

Exercise 5
(a) $2-8\sqrt{2}$  (b) $-5-2\sqrt{3}$  (c) $11+2\sqrt{10}$  (d) 2

Exercise 6
(a) $\sqrt{2}$  (b) 4  (c) 6  (d) 2

Exercise 7
1. (a) $\dfrac{\sqrt{10}}{2}$  (b) $\dfrac{\sqrt{10}}{10}$  (c) 3

2. (a) $\dfrac{2\sqrt{2}+\sqrt{6}}{2}$  (b) $\sqrt{3}+\sqrt{2}$  (c) $-7-4\sqrt{3}$

Exercise 8
$4\sqrt{3}+7$