RMIT UNIVERSITY

Applications of Differentiation

DN1.10: RATES OF CHANGE

If there is a relationship between two or more variables, for example,

area and radius of a circle where $A = \pi r^2$

or length of a side and volume of a cube where $V = \ell^3$

then there will also be a relationship between the rates at which they change.

> If y is a function of x ie y = f(x)
>
> then $f'(x) = \dfrac{dy}{dx}$
>
> is the rate of change of y with respect to x

We can use differentiation to find the function that defines the rate of change between variables

$$\frac{dA}{dr} = 2\pi r \quad \text{(differentiating with respect to r)}$$

and $$\frac{dV}{d\ell} = 3\ell^2 \quad \text{(differentiating with respect to } \ell\text{)}$$

The chain rule can be used to find rates of change with respect to time:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

So that when $A = \pi r^2$:
$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 2\pi r \frac{dr}{dt}$$

and when $V = \ell^3$:
$$\frac{dV}{dt} = \frac{dV}{d\ell} \times \frac{d\ell}{dt} = 3\ell^2 \frac{d\ell}{dt}$$

Examples

1. A balloon has a small hole and its volume V (cm^3) at time t (sec) is $V = 66 - 10t - 0.01t^2$, $t > 0$
   Find the rate of change of volume after 10 seconds.

$$V = 66 - 10t - 0.01t^2$$

$$\frac{dV}{dt} = -10 - 0.02t$$

When $t = 10$, $\frac{dV}{dt} = -10 - (0.02)(10)$

$$= -10.2 \ cm^3/sec$$

2. The pressure P , of a given mass of gas kept at constant temperature, and its volume V are connected by the equation PV = 500.

Find $\frac{dP}{dV}$ *when V = 20.*

$$PV = 500$$

ie. $P = \frac{500}{V}$

ie. $P = 500V^{-1}$

Then $\frac{dP}{dV} = -500V^{-2}$

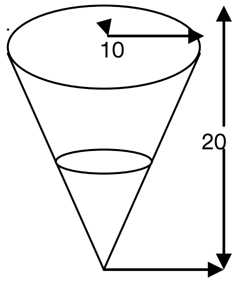
When V = 20: $\frac{dP}{dV} = -500(20)^{-2}$

$$= -1.25$$

3. Water is running out of a conical funnel at the rate of $5cm^3/s$. The radius of the funnel is 10 cm and the height is 20 cm. How fast is the water level dropping when the water is 10 cm deep?

Let *h* be the depth, *r* the radius and *V* be the volume of the water at time t

Then $\frac{dV}{dt} = -5$ (since the rate is decreasing)



By similar triangles:

$$\frac{r}{h} = \frac{10}{20}$$

$$r = \frac{h}{2}$$

$V = \frac{1}{3}\pi r^2 h$ [formula for volume of a cone]

$$= \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \times \frac{dh}{dt}$$

When $h = 10$,

$$-5 = \frac{\pi}{4} \times 100 \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{1}{5\pi}$$

The water level is dropping at a rate of $-\frac{1}{5\pi} cm/s$

Exercises

1. The radius of a spherical balloon is increasing at a rate of 3 cm/min. At what rate is the volume increasing when the radius is 5cm?
2. If the displacement of an object from a starting point is given by s(t) = sin(t) – 2cos(t) find the velocity when t = 1.
   Hint: v(t) = s'(t)
3. The kinetic energy K of a body of mass m moving with speed v is given by $K = ½mv^2$. Find a formula for the instantaneous rate of change of energy with respect to velocity for a body with a mass of 10kg.
4. The function $n(t) = 200t - 100\sqrt{t}$ describes the spread of a virus where t is the number of days since the initial infection and n is the number of people infected. Find the rate at which n is increasing at the instant when $t = 4$.
5. The profit P made from selling a certain item is related to the number sold x by the formula $P(x) = 10000 - x^2 + 520x$. What is the rate of change of profit with respect to number sold when x = 260? What is the significance of this answer?
6. If $y = \left(x - \frac{1}{x}\right)^2$ find $\frac{dx}{dt}$ when x = 2, given $\frac{dy}{dt} = 1$.
7. A hollow right circular cone is held vertex downwards beneath a tap leaking at the rate of $2cm^3/s$. Find the rate of rise of water level when the depth is 6cm given that the height of the cone is 18 cm and its radius 12cm.
8. The area of a circle is increasing at the rate of $3cm^2/s$. Find the rate of change of the circumference when the radius is 2cm.

Answers

1. $300\pi \cong 942$ cm^3/min
2. 2.22
3. $\frac{dK}{dv} = 10v$
4. 175
5. 0 which means this is the point at which profit is maximized.
6. $\frac{4}{15}$
7. $\frac{1}{8\pi}$ cm/s $\cong$ 0.04 cm/s
8. 1.5 cm/s