

AF1.4: PARTIAL FRACTIONS

Adding fractions

To add fractions, rewrite the fractions with a common denominator then add the numerators.

Example

Find the sum of

$$\frac{3}{3x+4} + \frac{2}{x-5}$$

The common denominator of $\frac{3}{3x+4}$ and $\frac{2}{x-5}$ is $(3x+4)(x-5)$

$$\frac{3(x-5)}{(3x+4)(x-5)} + \frac{2(3x+4)}{(3x+4)(x-5)} = \frac{3(x-5) + 2(3x+4)}{(3x+4)(x-5)}$$

$$= \frac{9x-7}{3x^2-11x-20}$$

$$\frac{3}{3x+4} + \frac{2}{x-5} = \frac{9x-7}{3x^2-11x-20}$$

The reverse of this process is to split a fraction into partial fractions. In the above example

$$\frac{9x-7}{3x^2-11x-20} = \frac{3}{3x+4} + \frac{2}{x-5}$$

Algebraic fraction

Partial fractions

If the degree of the numerator of the algebraic fraction is greater than that of the denominator, divide the denominator into the numerator then express the remaining fractional part as partial fractions

The form of the numerator in the partial fractions depends only on the type of the factors in the denominator of the original fraction, as indicated below.

Each distinct linear factor eg. $(x-a)$ has a corresponding partial fraction of the form	$\frac{A}{(x-a)}$ where A is a constant
Each repeated linear factor eg. $(x-a)^2$ has a corresponding partial fractions of the form	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$ A and B constants.
Each quadratic factor eg. ax^2+bx+c has a corresponding partial fraction of the form.	$\frac{Ax+B}{ax^2+bx+c}$ A and B constants.

See Exercise 1

Finding partial fractions

In general the numerator of a partial fraction is a polynomial of degree one less than the factor in the denominator. *Note the special case of repeated factors, example (2) above.*

To express an algebraic fraction as partial fractions:

1. factorise the denominator.
2. write the algebraic fraction in partial fraction form
3. add the partial fractions
4. determine the value of the constants.

Linear factors

Example

Express $\frac{x-5}{x^2+2x-3}$ as the sum of partial fractions.

$$\frac{x-5}{x^2+2x-3} = \frac{x-5}{(x-1)(x+3)} \quad \text{factorise the denominator}$$

$$\frac{x-5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

Each linear factor in the denominator factor has one corresponding partial fraction, a constant as the numerator.

$$\frac{x-5}{(x-1)(x+3)} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

add the fractions on the right-hand side of the equation

$$x-5 = A(x+3) + B(x-1)$$

equate the numerators

$$x-5 = (A+B)x + (3A-B)$$

expand brackets, collect like terms and equate coefficients of powers of .x

$$\begin{aligned} A+B &= 1 && \text{coefficients of } x \\ 3A-B &= -5 && \text{constant terms} \end{aligned}$$

Solve these simultaneous equations to give

$$A = -1 \quad \text{and} \quad B = 2$$

Substituting for A and B in the partial fractions gives

$$\frac{x-5}{x^2+2x-3} = \frac{2}{x+3} - \frac{1}{x-1}$$

Solving the simultaneous equations that result from equating coefficients can sometimes be quite lengthy. An alternative method is to equate the numerators and, before expanding the brackets, substitute a value of x into both sides of the equation so that only one variable remains. Repeat this to find other variables. This method will not necessarily find all variables, but will often make calculations easier.

Using the previous example, after equating numerators

$$x - 5 = A(x + 3) + B(x - 1)$$

Substituting $x = 1$ will eliminate B and substituting $x = -3$ will eliminate A .

if $x = 1$ then

$$1 - 5 = A(1 + 3) + B(1 - 1)$$

solving gives $A = -1$ as before.

if $x = -3$ then

$$-3 - 5 = A(-3 + 3) + B(-3 - 1)$$

solving gives $B = 2$ as before.

Repeated linear factors

Example

Express $\frac{5x^2 + 3x + 1}{x^3 - 3x - 2}$ as partial fractions.

$$\frac{5x^2 + 3x + 1}{x^3 - 3x - 2} = \frac{5x^2 + 3x + 1}{(x + 1)^2(x - 2)}$$

factorise the denominator

$$\frac{5x^2 + 3x + 1}{x^3 - 3x - 2} = \frac{A}{(x - 2)} + \frac{B}{(x + 1)} + \frac{C}{(x + 1)^2}$$

The repeated linear factor, $(x + 1)^2$, has one corresponding partial fraction with denominator $(x + 1)$ and one with denominator $(x + 1)^2$. The factor $(x - 2)$ has one corresponding partial fraction. All numerators are a constant.

$$\frac{5x^2 + 3x + 1}{x^3 - 3x - 2} = \frac{A(x + 1)^2 + B(x + 1)(x - 2) + C(x - 2)}{(x - 2)(x + 1)^2}$$

add the fractions on the right-hand side of the equation

$$5x^2 + 3x + 1 = A(x + 1)^2 + B(x + 1)(x - 2) + C(x - 2)$$

equate the numerators

Substitute $x = 2$ (terms involving A and B will be zero).

$$5 \times 2^2 + 3 \times 2 + 1 = A(2 + 1)^2 + B(2 + 1)(2 - 2) + C(2 - 2) \Rightarrow A = 3$$

Substitute $x = -1$ (terms involving A and B will be zero).

$$5 \times (-1)^2 + 3 \times (-1) + 1 = A(0)^2 + B(0)(-3) + C(-3) \Rightarrow C = -1$$

Substitute $x = 0$ and using $A = 3$ and $C = -1$ (as found)

$$0 + 0 + 1 = 3(1)^2 + B(1)(-2) - 1(-2) \Rightarrow B = 2$$

Constants are $A = 3$, $B = 2$, and $C = -1$

Substituting for A and B and C in the partial fractions gives

$$\frac{5x^2 + 3x + 1}{x^3 - 3x - 2} = \frac{3}{(x - 2)} + \frac{2}{(x + 1)} - \frac{1}{(x + 1)^2}$$

Quadratic or higher factor

The numerator for a quadratic factor has the form $Ax+B$. In general if the denominator is of degree n then the numerator of the partial fraction is a polynomial of degree $n-1$.

Example

Express $\frac{5x^2 - 6x + 5}{x^3 - 2x^2 + 3x - 2}$ in terms of partial fractions.

$$\frac{5x^2 - 6x + 5}{x^3 - 2x^2 + 3x - 2} = \frac{5x^2 - 6x + 5}{(x-1)(x^2 - x + 2)}$$

factorise the denominator

$$\frac{5x^2 - 6x + 5}{x^3 - 2x^2 + 3x - 2} = \frac{A}{(x-1)} + \frac{Bx + C}{(x^2 - x + 2)}$$

The factor $(x-1)$ has one corresponding partial fraction, with a constant as the numerator. The quadratic factor has one corresponding partial fraction with a linear denominator.

$$\frac{5x^2 - 6x + 5}{x^3 - 2x^2 + 3x - 2} = \frac{A(x^2 - x + 2) + (Bx + C)(x-1)}{(x-1)(x^2 - x + 2)}$$

add the fractions on the right side

$$5x^2 - 6x + 5 = A(x^2 - x + 2) + (Bx + C)(x-1)$$

equate numerators

$$\begin{aligned} 5x^2 - 6x + 5 &= Ax^2 - Ax + 2A + Bx^2 - Bx + Cx - C \\ &= (A+B)x^2 + (C-A-B)x + (2A-C) \end{aligned}$$

expand the brackets on the right-hand side of the equation and collect like terms

$$A + B = 5$$

$$C - A - B = -6$$

equate coefficients of powers of x

$$2A - C = 5$$

Solve these simultaneous equations to give $A = 2$, $B = 3$, and $C = -1$

Substituting for A and B in the partial fractions gives

$$\frac{5x^2 - 6x + 5}{x^3 - 2x^2 + 3x - 2} = \frac{2}{(x-1)} + \frac{3x-1}{x^2 - x + 2}$$

See Exercise 2

Exercises

1. Write the following in partial fraction form, but **do not** calculate the numerical values for the constants in the numerator. If possible, factorise the quadratic factor first.

$$(a) \frac{x+6}{(2x-3)(x+4)} \quad (b) \frac{2x^2+x-3}{(x)(x+1)(x-1)(x-2)} \quad (c) \frac{2x}{(x^2+3)(x+1)}$$

$$(d) \frac{x^2-x+3}{(x^2-3x+7)(x+2)^2}$$

2. Express the following as partial fractions.

$$(a) \frac{2x}{(x-3)(x+2)} \quad (b) \frac{x+2}{x^2-5x+6} \quad (c) \frac{x^2+2x+4}{(x+1)(x^2+3x+3)}$$

$$(d) \frac{3x}{(1-x)^2} \quad (e) \frac{x^2-2x+2}{x^3+x^2+x}$$

Answers

$$1. (a) \frac{A}{2x-3} + \frac{B}{x+4} \quad (b) \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} + \frac{D}{x-2} \quad (c) \frac{A}{x+1} + \frac{Bx+C}{x^2+3}$$

$$(d) \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2-3x+7}$$

$$2. (a) \frac{6}{5(x-3)} + \frac{4}{5(x+2)} \quad (b) \frac{5}{x-3} - \frac{4}{x-2} \quad (c) \frac{3}{(x+1)} - \frac{2x+5}{x^2+3x+3}$$

$$(d) \frac{3}{(1-x)^2} - \frac{3}{(1-x)} \quad (e) \frac{2}{x} - \frac{x+4}{x^2+x+1}$$