RMIT UNIVERSITY

DN1.12: IMPLICIT DIFFERENTIATION

Derivative of an implicit function

If we are able to put an equation relating x and y in the form of $y = f(x)$ then we can find the derivative $\frac{dy}{dx}$ or y'. Note: $\frac{dy}{dx}$ and y' mean the same thing.

Example 1

$$y - 3x^2 + 2x - 5 = 0 \quad \Rightarrow y = 3x^2 - 2x + 5$$

$$\frac{dy}{dx} = 6x - 2 \quad \text{or} \quad \frac{d}{dx}(3x^2 - 2x + 5) = 6x - 2$$

But what about $\frac{d}{dx}(y^5 + 3xy + x^2 - 5)$ or similar? These are much harder to differentiate.

Example 2

Find $\frac{d}{dx}(y^2 x)$

Using the Product Rule first

$$\frac{d}{dx}(y^2 x) = y^2 \frac{d}{dx}(x) + x\frac{d}{dx}(y^2) \qquad \text{Equation 1}$$

Now, using the Chain Rule for $\frac{d}{dx}(y^2)$

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Substituting back into Equation 1

$$\frac{d}{dx}(y^2 x) = y^2.(1) + x(2y)\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx}$$

Example 3

Chain Rule *Product Rule*

Find $\frac{d}{dx}(y^3 + 2xy)$

$$\frac{d}{dx}(y^3 + 2xy) = \frac{d}{dx}(y^3) + \frac{d}{dx}(2xy) \quad = \underbrace{\frac{d}{dy}(y^3)\frac{dy}{dx}} + \underbrace{2x\frac{d}{dx}(y) + y\frac{d}{dx}(2x)}$$

$$= 3y^2\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y$$

Exercise 1

Find

1. $\frac{d}{dx}\left(\frac{x}{y}\right)$

2. $\frac{d}{dx}\left(\frac{x^3}{y^3}\right)$

3. $\frac{d}{dx}\left(\frac{x+y}{x-y}\right)$

Finding the equation of tangent lines to implicit functions

Example 4

Find the equation of the tangent line $y^3 + 2xy = 6$ at the point (1, 1)

See Example 2 above

First, we need to find the slope y' of this curve at (1, 1).

$$\frac{d}{dx}(y^3 + 2xy = 6) \Rightarrow \frac{d}{dx}(y^3 + 2xy) = \frac{d}{dx}(6) \quad \Rightarrow \quad \underbrace{3y^2y' + 2xy' + 2y} = 0$$

Derivative of a constant is zero

$$\Rightarrow \quad y' = \frac{-2y}{3y^2 + 2x}$$

At (1, 1): $\quad y' = \frac{-2(1)}{3(1)^2 + 2(1)} = \frac{-2}{5}$

Since the equation of a line is $y - y_1 = m(x - x_1)$ where $m = y' = \frac{-2}{5}$, and $x_1 = 1$, $y_1 = 1$:

$$y - 1 = \frac{-2}{5}(x - 1) \qquad \Rightarrow \qquad 2x + 5y = 7$$

Hence, the equation of the tangent line is $2x + 5y = 7$

Exercise 2

1. Find the equation of the tangent line to the graph of $x^2 + 2xy + y = 7$ at the point (1, 2).
2. Find the equation of the tangent line to the graph of $x^2 + 4y^2 = 8$ at the point (2, -1).
3. Find the equation of the tangent line to the graph of $3xy - y^2 + x + 9 = 0$ at the point (1, -2).

Answers

Exercise 1

1. $\dfrac{y - xy'}{y^2}$ 2. $\dfrac{3x^2y^3 - 3x^3y^2y'}{y^6}$ 3. $\dfrac{2xy'}{(x-y)^2}$

Exercise 2

1. $y - 2 = -2(x - 1)$ 2. $y + 1 = \frac{1}{2}(x - 2)$ 3. $y + 2 = \frac{5}{7}(x - 1)$