

IL1.2: FRACTIONAL INDICES

Previously we considered integer indices. What does a fractional index mean?
The index laws apply to fractional indices as well as positive and negative integer indices.

Using the first index law we know that

$$3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3^1$$

That is $3^{\frac{1}{2}}$ multiplied by itself equals 3.

The square root of 3 is the number that, when multiplied by itself, equals 3 and is written as $\sqrt{3}$.

$$\sqrt{3} \times \sqrt{3} = 3^1$$

Since $3^{\frac{1}{2}}$ behaves like $\sqrt{3}$ we say that $3^{\frac{1}{2}} = \sqrt{3}$.

Similarly

$$\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2 \quad \text{and}$$

$$2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}} = 2^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 2 \quad \text{using the first index law}$$

Since $2^{\frac{1}{3}}$ behaves like the cube root $\sqrt[3]{2}$; we say $2^{\frac{1}{3}} = \sqrt[3]{2}$.
In general:

$a^{\frac{1}{n}} = \sqrt[n]{a} \quad (n^{\text{th}} \text{ root of } a)$	where n is a positive integer.
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Examples

(1). $4^{\frac{1}{2}} = \sqrt{4} = 2$

(2). $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

(3). $3^{\frac{1}{4}} = \sqrt[4]{3}$

(4). $b^{\frac{1}{5}} = \sqrt[5]{b}$

(5). $x^{\frac{1}{2}} = \sqrt{x}$

(6). $32^{-\frac{1}{5}} = \frac{1}{32^{\frac{1}{5}}} = \frac{1}{\sqrt[5]{32}} = \frac{1}{2}$

In most cases the root of a number will not be able to be written as a fraction and will be an irrational number.
For example $\sqrt{2} = 1.414\dots\dots$

See Exercise 1

Expressions of the form $a^{\frac{m}{n}}$

If $a^{\frac{1}{n}} = \sqrt[n]{a}$ (n^{th} root of a), what does $a^{\frac{2}{3}}$ mean?

$a^{\frac{2}{3}}$ can be written $(a^2)^{\frac{1}{3}}$ using the third index law

So $a^{\frac{2}{3}} = a^{\frac{2}{3}} = \sqrt[3]{a^2}$ using the definition of indices as roots

In general

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

where m and n are integers.

Examples

(1). $5^{\frac{3}{4}} = \sqrt[4]{5^3}$

(2). $7^{\frac{5}{2}} = \sqrt{7^5}$

(3). $a^{\frac{7}{5}} = \sqrt[5]{a^7}$

(4). $y^{-\frac{3}{4}} = \frac{1}{y^{\frac{3}{4}}} = \frac{1}{\sqrt[4]{y^3}}$

(5). $\sqrt[4]{x^3} = x^{\frac{3}{4}}$

The index laws can be used to evaluate and simplify expressions with fractional indices.

Examples

Simplify the following, and evaluate if possible.

(1) $(x^3 y^9)^{\frac{2}{3}}$; $(x^3 y^9)^{\frac{2}{3}} = x^{\frac{3 \times 2}{3}} y^{\frac{9 \times 2}{3}} = x^2 y^6$ using third index law

(2) $3^{\frac{1}{3}} \div 3^{\frac{4}{3}}$; $3^{\frac{1}{3}} \div 3^{\frac{4}{3}} = 3^{\frac{1}{3} - \frac{4}{3}} = 3^{-1} = \frac{1}{3}$ using second index law

(3) $32^{\frac{3}{5}}$; $32^{\frac{3}{5}} = (2^5)^{\frac{3}{5}} = 2^3 = 8$ write 32 as 2^5 . Use third index law.

(4) $25^{\frac{1}{2}}$; $25^{\frac{1}{2}} = \frac{1}{25^{\frac{1}{2}}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$ power $\frac{1}{2}$ - same as the square root.

(5) $(a^2 b^5)^{\frac{1}{3}} \times a^{\frac{1}{3}} b^{-\frac{2}{3}}$

$$\begin{aligned} (a^2 b^5)^{\frac{1}{3}} \times a^{\frac{1}{3}} b^{-\frac{2}{3}} &= a^{\frac{2}{3}} b^{\frac{5}{3}} \times a^{\frac{1}{3}} b^{-\frac{2}{3}} \\ &= a^{\frac{2}{3} + \frac{1}{3}} b^{\frac{5}{3} - \frac{2}{3}} \\ &= ab \end{aligned}$$

See Exercise 2

Exercises

Exercise 1

Evaluate the following expressions. If the answer is not exact give the decimal approximation to two decimal places.

(a) $64^{\frac{1}{2}}$ (b) $125^{\frac{1}{3}}$ (c) $36^{\frac{1}{4}}$

(d) $81^{-\frac{1}{2}}$ (e) $128^{-\frac{1}{7}}$ (f) $250^{\frac{1}{5}}$

Exercise 2

Simplify the following expressions. Give your answer in index notation with positive indices.

(a) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ (b) $\sqrt{5} \times \sqrt[3]{5} \times \sqrt[6]{5}$ (c) $\sqrt{a} \times \sqrt[4]{a} \times \sqrt[3]{a^2}$

(d) $(125a^6b)^{-\frac{1}{3}} \times b^{\frac{2}{3}}$ (e) $\frac{(2xy^3)^{\frac{1}{2}}}{2} \times \left(\frac{x^{\frac{3}{2}}}{y^2}\right)^4$ (f) $2^{\frac{5}{2}} - 2^{\frac{3}{2}}$

Answers

Exercise 1

(a) 8 (b) 5 (c) 2.45 (d) $\frac{1}{9}$ (e) $\frac{1}{2}$ (f) 3.02

Exercise 2

(a) $\frac{2^2}{3^2}$ (b) 5 (c) $a^{\frac{17}{12}}$ (d) $\frac{b^{\frac{1}{3}}}{5a^2}$ (e) $\frac{x^{\frac{13}{2}}}{2^{\frac{1}{2}}y^{\frac{13}{2}}}$ (f) $2^{\frac{3}{2}}$