FA1.2: FACTORISATION: QUADRATICS

Definition

The general form of a quadratic expression is:

$$ax^2 + bx + c \qquad a \neq 0$$

where *a*, *b*, *c* are constants and *x* is the unknown.

Expansion

Revision of expansion.

To expand an expression of the form $(a+b)(c+d)$, multiply each term in the first bracket by each term in the second bracket.

$$(x+2)(x+3) = x(x+3) + 2(x+3) = x^2 + 3x + 2x + 6$$

Simplify by combining like terms :

$$(x+2)(x+3) = x^2 + 5x + 6$$

Factorisation

Factorisation is the reverse of expansion.

$x^2 + 5x + 6$ is expressed as the product of two factors, $(x+2)$ and $(x+3)$.

factorisation

$$x^2 + 5x + 6 = (x+2)(x+3)$$

expansion

Note that:

- multiplying the first term in each bracket gives the term 'x^2' in the expression
- multiplying the last term in each bracket gives the constant term, (+6), in the expression
- the coefficient of the 'x' term is the sum of last term in each bracket (+2 + 3 = +5).

In general

constant 'c' positive

$$x^2 + bx + c = (x+m)(x+n) \quad \text{and} \quad x^2 - bx + c = (x-m)(x-n)$$

constant 'c' negative

$$x^2 + bx - c = (x+m)(x-n) \qquad m > n$$

$$x^2 - bx - c = (x+m)(x-n) \qquad m < n$$ where m and n are ≥ 0.

Expressions with *a = 1*

If $a = 1$ the expression becomes $\boldsymbol{x^2 + bx + c}$

To factorise:

> Find two numbers that:
>
> - multiply to '*c*' and
> - add to '*b*'
>
> Insert one number into each of the brackets (x ….)(x ….).
> The order is not important.

Note:

- If 'c' is positive, both numbers are positive or both numbers are negative.
  If b is positive, both numbers are positive.
  If b is negative, both numbers are negative.

- If c is negative, then one number is positive and the other negative;
  If b is positive, then the larger number is positive.
  If b is negative, then the larger number is negative.

Examples

Factorise the following.

1. $x^2 + 9x + 14$

$$x^2 + 9x + 14 = (x\ldots\quad)(x\ldots\quad)$$
$$= (x + 2)(x + 7)$$

the first term in each bracket must be 'x'

Look for two numbers that

- multiply to +14
- add to +9

the numbers are +2 and +7

2. $y^2 - 7y + 12$

$$y^2 - 7y + 12 = (y\ \ldots.)(y\ \ldots\ )$$
$$= (y - 3)(y - 4)$$

the first term in each bracket must be 'y'

Look for two numbers that

- multiply to +12.
- add to – 7,

the numbers are – 3 and – 4

3. $p^2 - 5p - 14$

$$p^2 - 5p - 14 = (p\ldots\quad)(p\ldots.\quad)$$
$$= (p - 7)(p + 2)$$

the first term in each bracket must be 'p'

Look for two numbers that

- multiply to – 14
- add to – 5,

(the larger number must be – ve)
numbers are – 7 and +2

4. $a^2 + 6a - 7$

$$a^2 + 6a - 7 = (a\ldots\quad)(\ a\ldots\quad)$$
$$= (a + 7)(\ a - 1\ )$$

the first term in each bracket must be 'a'

Look for two numbers that

- multiply to – 7
- add to + 6

(the larger number must be +ve)
numbers are + 7 and –1

Not all expressions of the form $x^2 + bx + c$ can be factorised by this method. The factors may be of a more complicated form or there may be no real factors.

Discriminant

For a quadratic equation of the form $ax^2 + bx + c$ a method of checking for real factors is to calculate the value of $(b^2 - 4ac)$ (called the ***discriminant***).

If $b^2 - 4ac < 0$ there are no real factors

Example

Factorise (if possible) $a^2 + 6a + 12$

In this case:
'b' = +6 'c' = +12
$\therefore$ $(b^2 - 4ac) = -12$
The discriminant is negative therefore $a^2 + 6a + 12$ has no real factors.

Exercise

Factorise the following (if possible).

1. $x^2 + 10x + 21$
2. $z^2 + 11z + 18$
3. $x^2 + 5x - 14$
4. $m^2 - m - 72$
5. $x^2 + 6x + 9$
6. $a^2 - 15a + 44$
7. $x^2 - 2x - 24$
8. $y^2 - 10y + 16$
9. $z^2 + 4z - 60$
10. $n^2 + 6n - 16$
11. $a^2 + 5a + 10$
12. $s^2 + 2s - 48$
13. $y^2 + 7y + 19$
14. $x^2 + 16x + 39$
15. $x^2 - 14x + 45$

Answers

1. $(x+7)(x+3)$
2. $(z+9)(z+2)$
3. $(x+7)(x-2)$
4. $(m-9)(m+8)$
5. $(x+3)(x+3)$
6. $(a-11)(a-4)$
7. $(x-6)(x+4)$
8. $(y-2)(y-8)$
9. $(z-6)(z+10)$
10. $(n-2)(n+8)$
11. no real factors
12. $(s+8)(s-6)$
13. no real factors
14. $(x+13)(x+3)$
15. $(x-9)(x-5)$

Expressions with a ≠ 1

Expressions of the type ax^2+bx+c can be factorised using a technique similar to that used for expressions of the type x^2+bx+c.

In this case the coefficient of x, in at least one bracket, will not equal 1.

Consider the following product.

$$\begin{aligned}(3x+2)(2x+1) &= 6x^2+3x+4x+2\\ &= 6x^2+7x+2\end{aligned}$$

- multiplying the first term in each bracket gives the 'x^2' term ($6x^2$) in the expansion
- multiplying the last term in each bracket gives the constant term (+2), in the expansion
- the coefficient of the 'x' term is the sum of the 'x' terms in the expansion. ($+3x+4x=+7x$).

Examples

1. Factorise $2x^2+7x+6$

$2x^2+7x+6 = (2x\ldots\quad)(x\ldots\quad)$

$2x^2+7x+6 = (2x+3)(x+2)$

- $2x \times x = 2x^2$
- find factors of + 6
  middle term positive so
  + 6 and +1 or +3 and +2
- try all combinations.
- the one that gives +7x when the brackets are expanded is $(2x+3)$ and $(x+2)$

2. Factorise $2x^2-x-6$

$2x^2-x-6 = (2x\ldots\quad)(x\ldots\quad)$

$2x^2-x-6 = (2x+3)(x-2)$

- $2x \times x = 2x^2$
- find factors of – 6.
- +6 & –1 or –6 & +1 or +3 & –2 or –3 & +2
- try all combinations
- the one that gives – x when the brackets are expanded is $(2x+3)$ and $(x-2)$

3. Factorise $3a^2-16a+5$

$3a^2-16a+5 = (3a\ldots\quad)(a\ldots\quad)$

$3a^2-16a+5 = (a-5)(3a-1)$

- $3a \times a = 3a^2$
- find factors of +5
- middle term negative so – 5 and – 1
- try each combination
- the one that gives – 16a when the brackets are expanded is $(a-5)$ and $(3a-1)$

4. Factorise $6y^2+16y-6$

$6y^2+16y-6 = 2(3y^2+8y-3)$

$= 2(3y\ldots\quad)(y\ldots\quad)$

$6y^2+16y-6 = 2(3y-1)(y+3)$

Remove a common factor of 2.
This reduces the number of possible combinations.

- $3y \times y = 3y^2$
- find factors of –3
- possibilities are +3 and – 1 or –3 and +1
- try all combinations
- the one that gives $+8y$ when the brackets are expanded is $(3y-1)$ and $(y+3)$

To factorise a quadratic of the form $ax^2 + bx + c$

- search each term in the expression for a common factor (**every term** must have this factor).
- remove this from the expression then place before the brackets.
- factorise, if possible, the resulting quadratic expression

This approach is good provided the number of combinations of factors of '*a*' and '*c*' is limited.
For more complicated expressions of the type $ax^2 + bx + c$ there are other methods of factorisation which may be more suitable.

Exercise

Factorise the following

(1) $5x^2 + 13x + 6$ (2) $2x^2 + x - 15$ (3) $3m^2 - m - 2$
(4) $3y^2 - 10y + 8$ (5) $2a^2 + 11a + 12$ (6) $6x^2 - 11x + 5$

Answers

(1) $(5x+3)(x+2)$ (2) $(2x-5)(x+3)$ (3) $(3m+2)(m-1)$
(4) $(3y-4)(y-2)$ (5) $(2a+3)(a+4)$ (6) $(6x-5)(x-1)$