FA1.3: FACTORISATION: COMPLETING THE SQUARE

A quadratic expression may be factorised by the method of "completing the square".

A **perfect square** is of the form

$$(x+b)^2 = x^2 + 2bx + b^2$$

The first two terms of the right-hand side of the above equation are ($x^2 + 2bx$). To make a perfect square or to **complete the square** on $x^2 + 2bx$,the third term, b^2 must be added.
Note that b^2 = (half the coefficient of x)^2.

Examples

To complete the square on $x^2 + 6x$ add (half of 6)^2, = $3^2 = 9$
$x^2 + 6x$ are the first two terms of the complete square $x^2 + 6x + 9$

To complete the square on $x^2 + 5x$ add (half of 5)^2, = $\left(\frac{5}{2}\right)^2$

$x^2 + 5x$ are the first two terms of the complete square $x^2 + 5x + \left(\frac{5}{2}\right)^2$

To complete the square on $x^2 - 10x$ add (half of 10)^2, = $5^2 = 25$

$x^2 - 10x$ are the first two terms of the complete square $x^2 - 10x + 25$

Expressions of the form $x^2 + bx + c$ can be factorised by completing the square on the first two terms and rewriting the expression in the form

$$\left[\left(x + \frac{b}{2}\right)^2 - m^2\right] \text{where } m^2 = -\left(\frac{b}{2}\right)^2 + c$$

The DOTS rule can then be used to factorise the expression.

Examples

Factorise by the method of completing the square

1. x^2+8x-5:

$$x^2+8x-5=\left(x^2+8x+16\right)-16-5$$

Halve the coefficient of x (8/2) =4
Square the result $4^2=16$

Add and subtract 16. (This does not change the expression)

$$=\left(x+4\right)^2-21$$

Simplify
$x^2+8x+16=(x+4)^2$
$-16-5=-21$

$$=\left(x+4\right)^2-\left(\sqrt{21}\right)^2$$

Write as the difference of two squares.

$$x^2+8x-5=\left(x+4+\sqrt{21}\right)\left(x+4-\sqrt{21}\right)$$

factorise using 'DOTS'

2. x^2-6x-4

Halve the coefficient of x. (6/2) = 3
Square the result $3^2=9$
Add and subtract 9

$$x^2-6x-4=\left(x^2-6x+9\right)-9-4$$

Simplify
$x^2-6x+9=(x-3)^2$, $-16-4=-21$

$$=\left(x-3\right)^2-13$$

$$=\left(x-3\right)^2-\left(\sqrt{13}\right)^2$$

Write as the difference of two squares.

$$x^2-6x-4=\left(x-3+\sqrt{13}\right)\left(x-3-\sqrt{13}\right)$$

factorise using 'DOTS'

See Exercise 1

If the coefficient of x^2 is not 1, remove the coefficient as a common factor and then factorise by completing the square.

3. $2x^2-10x+2$

$$2x^2-10x+2=2(x^2-5x+1)$$

Remove the common factor of 2

$$=2\left[\left(x^2-5x+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+1\right)\right]$$

Proceed as in example 1
Halve the coefficient of x
Square the result, then add and subtract the answer.

$$+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2$$

$$=2\left[\left(x-\frac{5}{2}\right)^2-\frac{25}{4}+\frac{4}{4}\right]$$

$$=2\left[\left(x-\frac{5}{2}\right)^2-\frac{21}{4}\right]$$

simplify

$$=2\left[\left(x-\frac{5}{2}\right)^2-\left(\frac{\sqrt{21}}{2}\right)^2\right]$$

write as difference of two squares

$$=2\left[\left(x-\frac{5}{2}+\frac{\sqrt{21}}{2}\right)\left(x-\frac{5}{2}-\frac{\sqrt{21}}{2}\right)\right]$$

factorise using DOTS

See Exercise 2

Exercises

Exercise 1
Factorise by completing the square (if possible).

(a) x^2+6x-5 (b) x^2+4x+2 (c) a^2-8a-2 (d) $x^2+6x+10$

(e) x^2-5x+5 (f) y^2+3y+4 (g) a^2-5a-1

Exercise 2
(a) $4x^2+8x-20$ (b) x^2-6x+2 (c) $12-4x-2x^2$

Answers

Exercise 1
(a). $\left(x+3+\sqrt{14}\right)\left(x+3-\sqrt{14}\right)$ (b) $\left(x+2+\sqrt{2}\right)\left(x+2-\sqrt{2}\right)$ (c) $\left(a-4+3\sqrt{2}\right)\left(x-4-3\sqrt{2}\right)$

(d) no real factors (e). $\left(x-\frac{5}{2}+\frac{\sqrt{5}}{2}\right)\left(x-\frac{5}{2}-\frac{\sqrt{5}}{2}\right)$ (f) no real factors (g) $\left(a-\frac{5}{2}+\frac{\sqrt{29}}{2}\right)\left(a-\frac{5}{2}-\frac{\sqrt{29}}{2}\right)$

Exercise 2
(a) $4\left(x+1+\sqrt{6}\right)\left(x+1-\sqrt{6}\right)$ (b) $\left(x-3+\sqrt{7}\right)\left(x-3-\sqrt{7}\right)$ (c) $-2\left(x+1+\sqrt{7}\right)\left(x+1-\sqrt{7}\right)$