RMIT UNIVERSITY

EXPONENTIAL EQUATIONS

Indicial (or exponential) equations have the form ax = b If we can write b as a number with a base a and an index, then we can equate the indices to find x.
If two equal numbers are written to the same base then the indices must be equal.

Examples

1. $3^x = 27$

   $3^x = 3^3$ 27 is written in index form, with a base of 3.

   $x = 3$ Each side of the equation is a now has a base of 3. Equate indices

2. $2^{1-x} = \frac{1}{8}$

   $2^{1-x} = 2^{-3}$ $\frac{1}{8}$ is written in index form with a base of 2.

   $1 - x = -3$
   $x = 4$ equate indices and solve for x.

3. $5^{\frac{1}{x}} = 25$

   $5^{\frac{1}{x}} = 5^2$ 25 is written in index form, with a base of 5.

   $x = \frac{1}{2}$. Equate indices and solve for x

See exercise 1

So far we have considered equations of the type: $3^x = 9$, where we obtain an integer solution for x.(x = 2)

Obviously this is not always the case; not all equations have integer solutions.

For example what does 3x = 10 imply for the value of x?

You can solve equations of this sort with your scientific calculator, which will give the logarithm of any number to base 10, and to base 'e' (Euler's number). Logarithms to base 'e' are often called natural logarithms.

On the calculator use the log button to evaluate logarithms to base 10 and the ln button to evaluate logarithms to base 'e'.

Examples

1. $3^x = 10$

Take logs (base 10) of both sides.

Then

$$\log_{10}\left(3^x\right) = \log_{10}(10)$$
$$x\log_{10}3 = 1$$
$$x = \frac{1}{\log_{10}3} \qquad \text{Use } \log_a x^y = y\log_a x \text{ to simplify}$$
$$x = \frac{1}{0.4771} \qquad \textit{(using calculator)}$$
$$x = 2.095$$

2. $2 \times 5^{x+1} = 15$

$5^{x+1} = 7.5$

Take logs of both sides.

$$\log_{10}5^{x+1} = \log_{10}7.5$$
$$(x+1)\log_{10}5 = \log_{10}7.5$$
$$(x+1) = \frac{\log_{10}7.5}{\log_{10}5} = 1.252 \qquad \text{use calculator}$$
$$x = 0.25$$

3. $2^{2x+1} = 5^{2-x}$ Take logs of both sides

$$\log 2^{2x+1} = \log 5^{2-x}$$
$$(2x+1)\log 2 = (2-x)\log 5$$
$$(2x+1)\frac{\log 2}{\log 5} = (2-x)$$
$$(2x+1)\times 0.43067 = (2-x)$$
$$1.86135x = 2 - 0.43067$$
$$x = 0.843$$ evaluate using calculator

See exercise 2

Growth and decay

Example

The number of bacteria present in a sample is given by

$N = 800e^{0.2t}$, where *t* is in seconds.

Find: (a) the initial number of bacteria
(b) the time when the number of bacteria reaches 10 000

(a) The initial number of bacteria occurs when $t = 0$

$N = 800e^{0.2t}$ Substitute t = 0 in the equation for N.

$$N = 800e^{0.2\times 0}$$
$$= 800e^{0}$$
$$N = 800$$

The initial number of bacteria is 800.

(b) The number of bacteria, *N,* is equal to 10 000.

$N = 800e^{0.2t}$ Substitute N = 10 000 in the equation for N.

$$10000 = 800e^{0.2t}$$
$$\frac{10000}{800} = e^{0.2t}$$
$$12.5 = e^{0.2t}$$
$$\ln 12.5 = 0.2t$$
$$t = 12.6$$ Take the logarithm to base e of both sides

It takes 12.6 sec. For the number of bacteria to reach 10 000.

Exercises

Exercise 1

Solve for x:

(a) $2^x = 8$ (b) $5^x = 125$ (c) $3^{x-1} = 27$ (e) $3^{2x-1} = 243$ (f) $3^{2x-1} = 243$

(g) $9^{\frac{1}{x}} = 3^{-4}$ (h) $9^{2x+1} = 27^x$

Exercise 2

Solve for the unknown, giving your answer to three decimal places.

(a) $5^x = 12$ (b) $2^{x-3} = 9$ (c) $4^{2x+1} = 5$ (d) $4^{x+1} = 5^x$ (e) $7^{-2x} = 5$

(f) $\frac{1}{2^{x+1}} = 4^{x+2}$

Exercise 3

(a) The decay rate for a radio-active element is given by

$$R = 400e^{-0.03t}$$

where R is the decay rate in counts/s at time t(s).

Find:

(i) the initial decay rate
(ii) the time for the decay rate to reduce to half the initial decay rate.

(b) The charge Q units on a plate of a condenser t seconds after it starts to discharge is given by the formula

$$Q = Q_0 10^{-kt}$$

If the initial charge is 5076 units and Q = 1840 when t = 0.5s find:

(i) the value of k

(ii) the time needed for the charge to fall to 1000 units.

(iii) the charge after 2 sec.

Answers

Exercise 1

.(a) $x = 3$ (b) $x = 3$ (c) $x = 4$ (d) $x = 4$ (e) $x = 3$ (f) $x = 3/4$ (g) $x = -1/2$
(h) $x = -2$

$x = 3$

Exercise 2.

((a) x = 1.544   (b) x = 6.170   (c) x = 0.080   (d) x = 6.213   (e) x = −0.414

(f) x = −1.667

Exercise 3.

(a) (ii) Ro = 400   (ii) t = 23.1s

(b). (i) k = 0.881   (ii) t = 0.8s   (iii) Q = 87.8 units