

DN1.1: DIFFERENTIATION FROM FIRST PRINCIPLES

The process of finding the derivative function using the definition

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \quad h \neq 0$$

is called *differentiating from first principles*.

Examples

1. Differentiate x^2 from first principles.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && h \neq 0 \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2x+h)}{h} \\ &= \lim_{h \rightarrow 0} 2x + h \\ &= 2x \end{aligned}$$

\therefore If $f(x) = x^2$ then $f'(x) = 2x$

2. Determine, from first principles, the gradient function for the curve

$$f(x) = 2x^2 - x \text{ and calculate its value at } x = 3$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, h \neq 0 \\ &= \lim_{h \rightarrow 0} \frac{2(x+h)^2 - (x+h) - [2x^2 - x]}{h} \\ &= \lim_{h \rightarrow 0} \frac{2x^2 + 4xh + 2h^2 - x - h - 2x^2 + x}{h} \\ &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 - h}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(4x + 2h - 1)}{h} \\ &= \lim_{h \rightarrow 0} 4x + 2h - 1 \\ &= 4x - 1 \end{aligned}$$

\therefore The gradient function is $f'(x) = 4x - 1$.

$$\text{At } x = 3, \quad f'(x) = 4x - 1 = 4(3) - 1 = 11$$

3. Use differentiation from first principles to find the gradient function of $y = \frac{1}{x}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad h \neq 0 \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} \times \frac{x}{x} - \frac{1}{x} \times \frac{x+h}{x+h}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x - x - h}{(x)(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{(x)(x+h)} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{(x)(x+h)} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{-1}{(x)(x+h)} \right) \\ &= \frac{-1}{x^2} \end{aligned}$$

Exercises

Find the derivative of the following, using differentiation from first principles.

1) $f(x) = 3x$

2) $f(x) = 5x^2 - 4$

3) $f(x) = 3x^2 - 2x$

4) $f(x) = x^2 + 6x + 9$

5) $f(x) = 2x^2 - \frac{1}{x}$

6) $f(x) = 10$

Answers

1) 3 2) $10x$ 3) $6x - 2$ 4) $2x + 6$ 5) $4x + \frac{1}{x^2}$ 6) 0