RMIT UNIVERSITY

CN1.2: POLAR FORM OF A COMPLEX NUMBER

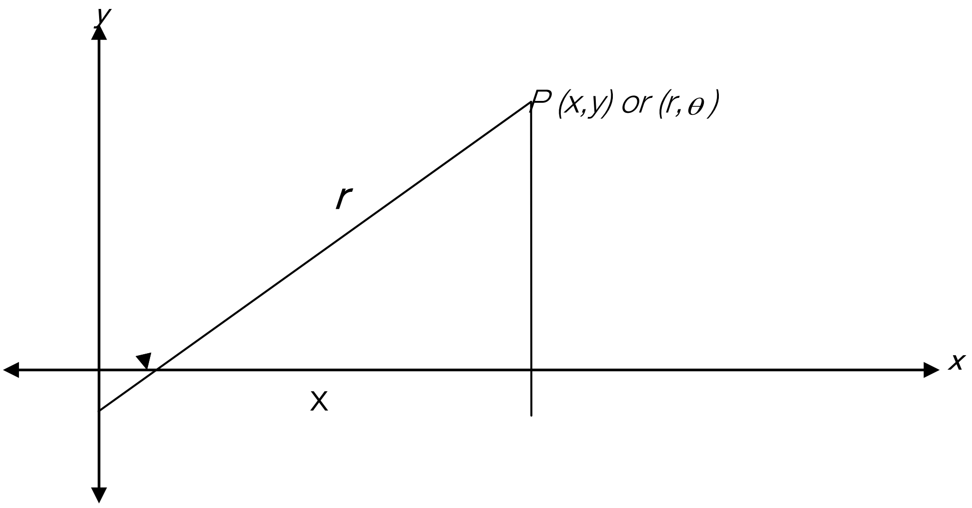
Rectangular and Polar Form

When a complex number is expressed in the form

$$z = x + yi$$

it is said to be in *rectangular form.*

But a point P with Cartesian coordinates (x,y) can also be represented by the polar coordinates (r, θ) where r is the distance of the point P from the origin and θ is the angle that $\overrightarrow{OP}$ makes with the positive x-axis



NB: $x = r\cos\theta$ and $y = r\sin\theta$

and $x^2 + y^2 = r^2$ or $r = \sqrt{x^2 + y^2}$

To express a complex number z in polar form:

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + r\sin\theta\, i \\ &= r(\cos\theta + \sin\theta\, i) \end{aligned}$$

which we abbreviate to $z = r\text{cis}\ \theta$

So, the polar form of the complex number z is

$$z = r\ cis\ \theta$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Modulus of z

The *modulus* of z, $|z|$ is the distance of the point z from the origin.

$$\text{mod } z = |z| = |x+yi| = \sqrt{x^2+y^2} = r$$

The *argument* of z, arg z, is the angle measured from the positive direction of the x-axis to $\overrightarrow{OP}$

If arg z = θ then sin $\theta = \frac{y}{|z|}$ and cos $\theta = \frac{x}{|z|}$ and tan $\theta = \frac{y}{x}$

An infinite number of arguments of z exist

eg If z = *i* then arg z = $\frac{\pi}{2} + 2\pi n, n \in Z$.

argument and Argument of z

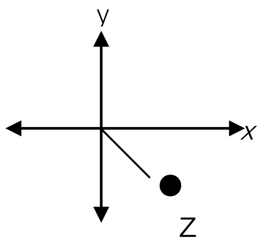
We define the Argument of z:

Arg z = θ, where $-\pi \le \theta \le \pi$

Examples

1. *Express in polar form z = 1 – i*

x = 1, y = -1 [NB: z is in the 4th quadrant]



r = $|z| = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$

tan $\theta = \frac{y}{x} = \frac{-1}{1} = -1$

$\theta = \tan^{-1}(-1) = \frac{-\pi}{4}$ [since z is in the 4th quadrant]

$\therefore z = r cis\theta$

$= \sqrt{2} \text{ cis} \left(\frac{-\pi}{4}\right)$

2. *Express 2 cis* $\left(\frac{4\pi}{3}\right)$ *in the form x + yi*

$2 \text{ cis}\left(\frac{4\pi}{3}\right) = 2\left[\cos\left(\frac{4\pi}{3}\right) + \sin\left(\frac{4\pi}{3}\right)i\right]$

$= 2 \times \left(-\frac{1}{2}\right) + 2 \times \left(-\frac{\sqrt{3}}{2}\right)i$

$= -1 - \sqrt{3}\,i$

See Exercise 1

Operations on Complex Numbers in Polar Form

Addition and Subtraction

Complex numbers in polar form are best converted to the form x + yi before addition or subtraction

Multiplication and Division

If $z_1 = r_1 cis\theta_1$ and $z_2 = r_2 cis\theta_2$ then it can be shown using trigonometric identities that

$$z_1 z_2 = r_1 r_2 cis(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$$

Examples:

1. If $z_1 = 2cis\frac{\pi}{4}$ and $z_2 = -3cis\frac{5\pi}{6}$ find $z_1 z_2$ in polar form, $-\pi \le \theta \le \pi$

$$z_1 z_2 = 2\text{cis}\frac{\pi}{4} \times \left(-3\text{cis}\frac{5\pi}{6}\right)$$

$$= -6\text{cis}\left(\frac{\pi}{4} + \frac{5\pi}{6}\right)$$

$$= -6\text{cis}\left(\frac{13\pi}{12}\right)$$

$$= -6\text{cis}\left(\frac{-11\pi}{12}\right) \quad \text{since } -\pi \le \theta \le \pi$$

2. If u = 1 + 3i and v = 2 – i find $\frac{u}{v}$ in polar form with $-\pi \le \theta \le \pi$

Two approaches are possible:

u = 1 + 3i ie. x = 1, y = 3

$r = \sqrt{1^2 + 3^2} = \sqrt{10}$

$\theta = \tan^{-1}\left(\frac{3}{1}\right) = 1.25^R$

$\therefore$ u = $\sqrt{10}$ cis1.25,

v = 2 – i ie x = 2, y = -1

$r = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$\theta = \tan^{-1}\left(\frac{-1}{2}\right) = -0.46^R$

$\therefore$ v = $\sqrt{5}$ cis0.46^R

Then $\frac{u}{v} = \frac{\sqrt{10}cis1.25^R}{\sqrt{5}cis0.46^R}$

$= \frac{\sqrt{10}}{\sqrt{5}}\text{cis}(1.25 + 0.46)^R$

$= \sqrt{2}\text{ cis } 1.71^R$

OR

$\frac{u}{v} = \frac{1+3i}{2-i}$

$= \frac{1+3i}{2-i} \times \frac{2+i}{2+i}$

$= \frac{-1+7i}{4+1}$

$= \frac{-1+7i}{5}$

$= -\frac{1}{5} + \frac{7}{5}i$

$\therefore x = -\frac{1}{5}, \quad y = \frac{7}{5}$

$r = \sqrt{x^2 + y^2} = \sqrt{(-0.2)^2 + 1.4^2} = \sqrt{2}$

$\tan\theta = \frac{1.4}{-0.2} = -7, \ \theta = 1.71^R$

$\therefore \frac{u}{v} = \sqrt{2}\text{ cis}(1.71)^R$

See Exercise 2

Exercises

Exercise 1

1. Find the polar form (in radians) of the following complex numbers:

(a) z = -1 + i　　(b) $z = -\sqrt{3} + i$

(c) z = - 3i　　(d) z = -2 – 4i

2. Express each of the following complex numbers in rectangular form

(a) $3\text{cis}\dfrac{\pi}{4}$　　(b) $\sqrt{7}cis\pi$

(c) $8\text{cis}\dfrac{\pi}{2}$　　(d) $10\text{cis}0.41^R$

3. If z = 2 + i and w = 1 – 4i find each of the following in polar form using radians where appropriate:

(a) $|z|$　　(b) $|w|$　　(c) Arg z　　(d) $|\overline{w}|$

(e) Arg(zw)　　(f) zw

Exercise 2

1. Simplify

(a) $4cis\dfrac{\pi}{3} \times 3cis\dfrac{\pi}{4}$　　(b) $\dfrac{3cis\dfrac{5\pi}{6}}{12cis\dfrac{\pi}{6}}$

2. If $u = 6\text{cis}\dfrac{3\pi}{4}$ and $v = 4\text{cis}\left(-\dfrac{\pi}{4}\right)$ express $\dfrac{u}{v}$ in polar form

3. If $z = 1 - \sqrt{3}i$, find $\overline{z}$ and express both z and $\overline{z}$ in polar form using radians.

Answers

Exercise 1

1. (a) $\sqrt{2}cis\dfrac{3\pi}{4}$　　(b) $2cis\dfrac{5\pi}{6}$

(c) $3cis\dfrac{-\pi}{2}$　　(d) $\sqrt{20}cis(-2.03)^R$

2. (a) $\dfrac{3}{\sqrt{2}} + \dfrac{3}{\sqrt{2}}i$　　(b) $-\sqrt{7}$

(c) 8i　　(d) 9.2 + 4i

3. (a) $\sqrt{5}$　　(b) $\sqrt{17}$　　(c) 0.46^R

(d) $\sqrt{17}$　　(e) -0.86^R　　(f) $9.22\text{cis}(-0.86)^R$

Exercise 2

1. (a) $12\text{cis}\dfrac{7\pi}{12}$　　(b) $\dfrac{1}{4}\text{cis}\dfrac{2\pi}{3}$

2. (a) $\dfrac{3}{2}cis\pi$

3. $z = 2cis\left(-\dfrac{\pi}{3}\right)$　　$\overline{z} = 2cis\left(\dfrac{\pi}{3}\right)$