CN1.1: COMPLEX NUMBERS

Real and complex numbers

Equations such as $x + 1 = 7$, $3x = 10$ and $x^2 - 7 = 0$ can all be solved within the real number system.

But there is no real number which satisfies $x^2 + 1 = 0$. To obtain solutions to this and other similar equations the ***complex numbers*** were developed.

The ***imaginary*** number i is defined such that $i^2 = -1$

ie and

$$i = \sqrt{-1}$$

NB: $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$ etc

A number z of the form $z = x + yi$ where x and y are real numbers is called a ***complex number***

x is called the ***real part*** of z, denoted by ***Re z***, and
y is called the ***imaginary part*** of z, denoted by ***Im z***

Examples

1. If $z = 5 - 3i$ then $\text{Re } z = 5$ and $\text{Im } z = -3$
2. If $z = \sqrt{3}\,i$ then $\text{Re } z = 0$ and $\text{Im } z = \sqrt{3}$

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal ie

$$a + bi = c + di$$

if and only if

$$a = c \text{ and } b = d$$

Example

If $z_1 = x - \frac{i}{3}$, $z_2 = \sqrt{2} + yi$ *and* $z_1 = z_2$ *find the values of x and y.*

$$\text{Re } z_1 = \text{Re } z_2 \Rightarrow x = \sqrt{2}$$

$$\text{and } \text{Im } z_1 = \text{Im } z_2 \Rightarrow y = -\frac{1}{3}$$

$$\therefore x = \sqrt{2} \text{ and } y = -\frac{1}{3}$$

Addition and Subtraction of Complex Numbers

To add or subtract complex numbers we add or subtract the real and imaginary parts separately:

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$$

Examples

1. $(2 + 3i) + (4 - i) = (2 + 4) + (3 - 1)i$
   $= 6 + 2i$

2. If $z_1 = 1 - i$ *and* $z_2 = 3 - 5i$ *find* $z_1 - z_2$

$z_1 - z_2 = (1 - i) - (3 - 5i)$
$= (1 - 3) + (-1- (-5))i$
$= -2 + 4i$

See ***Exercise 1***

Multiplication of Complex Numbers

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers then

$k z_1 = k(a + bi)$
$= ka + kbi$

and

$z_1 z_2 = (a + bi)(c + di)$
$= ac + adi + bci + bdi^2$
$= (ac - bd) + (ad + bc)i$ [since $i^2 = -1$]

Examples

1. Expand and simplify $i(3 + 4i)$

$i(3 + 4i) = 3i + 4i^2$
$= -4 + 3i$

2. If $z_1 = 1 - i$ *and* $z_2 = 3 - 5i$ *find* $z_1 z_2$

$z_1 z_2 = (1 - i)(3 - 5i)$
$= 3 - 3i - 5i + 5i^2$
$= 3 - 8i - 5$
$= -2 - 8i$

See ***Exercise 2***

Complex Conjugates

A pair of complex numbers of the form a + bi and a – bi are called ***complex conjugates.***

If z = x + yi then the conjugate of z is denoted by $\overline{z}$ = x – yi

Eg: 2 + 3i and 2 – 3i are a conjugate pair

1 – i and 1 + i are a conjugate pair

-4i and 4i are a conjugate pair

NB:

> The product of a conjugate pair of complex numbers is a ***real*** number

Since z$\overline{z}$ = (x + yi)(x – yi) = $x^2 + y^2$

Some properties of conjugates:

If z_1 and z_2 represent two conjugate numbers then

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(ii) $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$

(iii) $\overline{\overline{z}} = z$

Examples

If z = 2 – i and w = -3 + 4i find

1. $\overline{z}$ *2.* $\overline{z}$ - $\overline{w}$ *3.* $\overline{z + w}$

1. $\overline{z}$ = 2 + i

2. $\overline{z}$ - $\overline{w}$ = 2 + i - (-3 – 4i)
= 2 + 3 + i + 4i
= 5 + 5i

3. $\overline{z + w}$ = $\overline{2 - i + (-3 + 4i)}$

= $\overline{-1 + 3i}$

= -1 – 3i

See *Exercise 3*

Division of complex numbers

If z_1 = a + bi and z_2 = c + di, then $\frac{z_1}{z_2} = \frac{a+bi}{c+di}$.

To express $\frac{z_1}{z_2}$ in the form x + yi we make use of the conjugate to '*realize*' the denominator.

Examples

1. Express $\frac{2-i}{1+3i}$ *in the form x + yi*

$$\frac{2-i}{1+3i} = \frac{2-i}{1+3i} \times \frac{1-3i}{1-3i}$$

$$= \frac{2-i-6i-3}{1+9}$$

$$= \frac{-1-7i}{10}$$

$$= -\frac{1}{10} - \frac{7}{10}i$$

Operations on fractions involving complex numbers follow the same rules as algebraic fractions

2. $\frac{i}{1-4i} + \frac{2}{3+i} = \frac{i}{1-4i} \cdot \frac{1+4i}{1+4i} + \frac{2}{3+i} \cdot \frac{3-i}{3-i}$ [to rationalize denominators]

$$= \frac{i-4}{1+16} + \frac{6-2i}{9+1}$$

$$= \frac{i-4}{17} + \frac{6-2i}{10}$$

$= \frac{i-4}{17} \cdot \frac{10}{10} + \frac{6-2i}{10} \cdot \frac{17}{17}$ [170 is a common denominator]

$$= \frac{10i-40}{170} + \frac{102-34i}{170}$$

$$= \frac{10i-40+102-34i}{170}$$

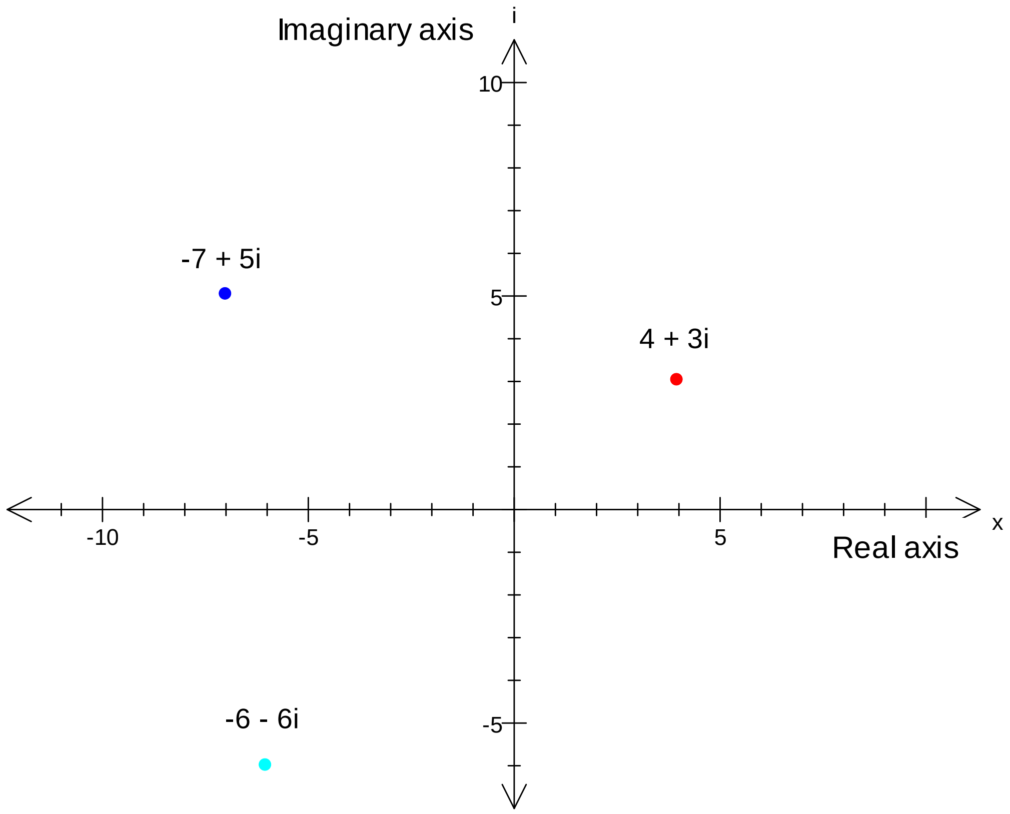
$$= \frac{62-24i}{170}$$

$$= \frac{2(31-12i)}{170}$$

$$= \frac{31-12i}{85}$$

See *Exercise 4*

An *Argand Diagram* is a geometrical representation of the set of complex numbers. The complex number $z = x + yi$ can plotted as a point represented by the ordered pair (x,y) on the complex number plane:



See *Exercise 5*

Exercises

Exercise 1

1. Express the following in terms of i in simplest surd form
   (a) $\sqrt{-9}$ (b) $\sqrt{-2}$ (c) $\sqrt{-5} \times \sqrt{3}$
   (d) $\sqrt{-5} \times \sqrt{10}$ (e) $\sqrt{-6} \times \sqrt{12}$

2. Evaluate
   (a) i^4 (b) i^9 (c) $i^7 - i^{11}$ (d) $i^5 + i^6 - i^7$ (e) $2i - i^6 + 2i^7$

3. State the value of *Re z* and *Im z* for these complex numbers:
   (a) 2 + 7i (b) 10 – i (c) π + 3i (d) $\frac{i}{6}$ (e) -8

4. Find the values of x and y
   (a) x + yi = 4 + 9i (b) x + yi = 3 – i
   (c) x + yi = 23 (d) x + yi = - $\sqrt{2}$ i
   (e) x + i = -5 + yi

Exercise 2

1. Expand and simplify
   (a) i(3 - 2i) (b) $2i^3(1 – 5i)$ (c) (8 – 3i)(2 – 5i)
   (d) $(4 – 3i)^2$ (e) (3 + 2i)(3 – 2i)

2. If z_1 = -1 + 3i and z_2 = 2 - i find each of the following
   (a) $z_1 z_2$ (b) $2z_1 - z_2$ (c) $(z_1 - z_2)^2$

3. Find the value of x and y if (x + yi)(2 – 3i) = -13i

Exercise 3

1. Find the conjugate of each of the following complex numbers:
   (a) 4 + 9i (b) -3-15i (c) $\sqrt{3}$ - 4i

2. Find the conjugate of (2 – i)(4 + 7i)

3. If z = 2 - i and w = 1 + 2i express the following in the form x + yi:
   (a) $\bar{z}$ (b) $\overline{z+w}$ (c) $\bar{z}$ + $\bar{w}$ (d) $\overline{zw}$ (e) $\overline{\bar{z}-\bar{w}}$

Exercise 4

1. Express the following in the form x + yi
   (a) $\frac{4-9i}{3}$ (b) $\frac{1}{3-i}$ (c) $\frac{5+i}{2-7i}$

2. Simplify $\frac{2}{1-i}+\frac{3+i}{i}$

3. If w = -1 + 6i express $\frac{w+1}{w-i}$ in the form x + yi

Exercise 5

If z = 2 – 3i and w = 1 + 4i, illustrate on an Argand diagram

1. z 2. w 3. z + w 4. $\overline{z+w}$ 5. 2z - w

Answers

Exercise 1

1. (a) 3i (b) $\sqrt{2}i$ (c) $\sqrt{15}i$ (d) $5\sqrt{2}i$ (e) $6\sqrt{2}i$

2. (a) 1 (b) i (c) 0 (d) 2i -1 (e) 1

3. (a) *Re z* = 2 *Im z* = 7 (b) *Re z* = 10 *Im z* = -1
   (c) *Re z* = π *Im z* = 3 (d) *Re z* = 0 *Im z* = $\frac{1}{6}$
   (e) *Re z* = *-8* *Im z* = 0

4. (a) x = 4, y = 9 (b) x = 3, y = -1
   (c) x = 23, y = 0 (d) x = 0, y = $-\sqrt{2}$
   (e) x = -5, y = 1

Exercise 2

1 (a) 2 + 3i (b) -10 - 2i (c) 1 – 46i (d) 7 – 24i (e) 13
2. (a) 1 + 7i (b) -4 + 7i (c) -7 – 24i
3 x = 3, y = -2

Exercise 3

1 (a) 4 – 9i (b) -3 + 15i (c) $\sqrt{3}$ + 4i
2. 15 – 10i
3. (a) 2 + i (b) 3 – i (c) 3 – i
   (d) 4 – 3i (e) 1 – 3i

Exercise 4

1 (a) $\frac{4}{3}$ - 3i (b) $\frac{3}{10}+\frac{1}{10}i$ (c) $\frac{3}{53}+\frac{37}{53}i$
2 2 – 2i
3 $\frac{15}{13}-\frac{3}{13}i$

Exercise 5