RMIT UNIVERSITY

DN1.5: CHAIN RULE

The **'chain rule'** is used to differentiate a function which is the *composition* of two simpler functions

If $y = g[u]$ where $u = h(x)$

Then
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Examples

1) *Differentiate* $y = (2x - 1)^4$

Let $u = 2x - 1$, then $y = u^4$

$$\frac{du}{dx} = 2 \quad \text{and} \quad \frac{dy}{du} = 4u^3$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 4u^3 \cdot 2$$

$$= 8u^3$$

$$= 8(2x - 1)^3 \qquad \text{[since } u = 2x - 1\text{]}$$

2) *Find the derivative of* $y = \dfrac{1}{\sqrt[3]{5t^2 + 2t + 1}}$

$$y = (5t^2 + 2t + 1)^{\frac{-1}{3}} \qquad \text{[change to index form for easier differentiation]}$$

Let $u = 5t^2 + 2t + 1$, then $y = \dfrac{1}{\sqrt[3]{u}} = u^{\frac{-1}{3}}$

$$\frac{du}{dt} = 10t + 2 \quad \text{and} \quad \frac{dy}{du} = \frac{-1}{3}u^{\frac{-4}{3}}$$

$$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt}$$

$$= \frac{-1}{3}u^{\frac{-4}{3}} \cdot (10t + 2)$$

$$= \frac{-1}{3}(5t^2 + 2t + 1)^{\frac{-4}{3}} \cdot (10t + 2) \qquad \text{[since } u = 5t^2 + 2t + 1\text{]}$$

$$= \frac{-(10t + 2)}{3}(5t^2 + 2t + 1)^{\frac{-4}{3}} \qquad \text{[after simplifying]}$$

3) *Differentiate y = sin5x*

y = sin5x

Let y = sin(u) where u = 5x

$\frac{dy}{du} = \cos(u)$ and $\frac{du}{dx} = 5$

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \cos(u).\ 5$

$= 5\cos(5x)$

4). *If $f(x) = \cos^3 x$ find $f'(x)$*

$y = \cos^3 x = [\cos(x)]^3$

Let $y = u^3$ where $u = \cos(x)$

$\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = -\sin(x)$

Then $f'(x) = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 3u^2.(-\sin x)$

$= 3\cos^2 x.(-\sin x)$

$= -3\sin x\cos^2 x$

5) *Differentiate $(\log_e 4x)^3$*

Let $y = u^3$ where $u = \log_e v$ and $v = 4x$ [The chain rule can be extended to three or more functions!!]

$\frac{dy}{du} = 3u^2,\ \frac{du}{dv} = \frac{1}{v}$ and $\frac{dv}{dx} = 4$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$

$= 3u^2.\frac{1}{v}.4$

$= 3(\log_e v)^2.\frac{1}{4x}.4$

$= 3(\log_e 4x)^2.\frac{1}{4x}.4$

$= \frac{3}{x}(\log_e 4x)^2$

Exercise

Find the derivatives of the following functions

1) y = $\tan 3x$

2) f(x) = $\log_e\left(\frac{x}{2}\right)$

3) y = $\sin\left(\frac{\pi}{4} - 2x\right)$

4) y = $\cos^2 x$

5) f(x) = $e^{\sin x}$

6) y = $\sqrt{1 - \cos 5x}$

Answers

1) $3\sec^2 3x$

2) $\frac{1}{x}$

3) $-2\cos\left(\frac{\pi}{4} - 2x\right)$

4) $-2\sin x \cos x$

5) $e^{\sin x}\cos x$

6) $\frac{5\sin 5x}{2\sqrt{1-\cos 5x}}$